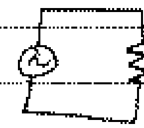
Optional

1st lets "drive" a resistor with an alternating voltage

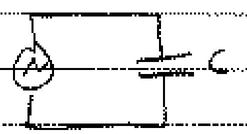
$$v = V\sin(\omega_0 t)$$



the current in the resistor is $I = \frac{V}{R}$

$$i(t) = \frac{V}{R}\sin(\omega_0 t)$$ no big surprise

now, lets try a capacitor



$$v = V\sin(\omega_0 t) \qquad q = CV$$

$$q(t) = CV\sin(\omega_0 t)$$

$$i(t) = \frac{dq(t)}{dt} = \omega_0 CV\cos(\omega_0 t)$$

sin like v

$$= (\omega_0 C)V\sin\left(\omega_0 t + \frac{\pi}{2}\right)$$

and to make this look like $V = IR$
were going to define something called capacitive reactance

$$X_C = \frac{1}{\omega_0 C}$$ has units $\frac{1}{\frac{1}{s}\cdot\frac{s}{\Omega}} = \Omega$!

so $$i(t) = \frac{V}{X_C}\sin\left(\omega_0 t + \frac{\pi}{2}\right)$$

How do we get this strange behavior?

well, we could look at the loop equation

at the start we have q leaving the cap
and current increasing

$$i = -\frac{dq_{cap}}{dt}$$

$$\frac{di}{dt} = -\frac{d^2 q_{cap}}{dt^2}$$

q_{cap} → or $\frac{di}{dt}$ is positive

i → $\frac{dq_{cap}}{dt}$ is negative

remember, we had to do something special
to get the signs right

$$\frac{q_{cap}}{C} = -L\frac{dI_q}{dt}$$

start here

$$\text{or} \quad \frac{d^2 q_{cap}}{dt^2}L + \frac{q_{cap}}{C} = 0$$

lets just make this simpler

$$\frac{d^2 q}{dt^2}L + \frac{q}{C} = 0$$

now, we're going to do a really nasty
trick $i\left(L\frac{di}{dt} + \frac{q}{C}\right) = 0\,(\;)$

$$Li\frac{di}{dt} + \frac{dq}{dt}\frac{q}{C} = 0$$

now integrate with respect to time

$$\int Li\frac{di}{dt}dt + \int \frac{1}{C}q\frac{dq}{dt} \Rightarrow L\int i\,di + \frac{1}{C}\int q\,dq$$

@ we'll combine the constants of integration

$$\frac{1}{2} L i^2 + \frac{1}{2}\frac{q^2}{C} = \text{Constant} = U$$

this constant is the energy of the system at $t=0$ $i=0$ $q=q_0$ ...

Let's keep going. Nasty trick #2

$$i^2 = \left(U - \frac{1}{2}\frac{q^2}{C}\right)\frac{2}{L}$$

$$\text{or} \quad \left(\frac{dq}{dt}\right)^2 = \left(\frac{2U}{L} - \frac{q^2}{LC}\right)$$

$$\frac{dq}{dt} = \sqrt{\frac{2U}{L} - \frac{q^2}{LC}}$$

$$\frac{dq}{\sqrt{\frac{2U}{L} - \frac{q^2}{LC}}} = dt$$

$x = a\sin\theta$

$$\frac{a\cos\theta\, d\theta}{a\sqrt{1-\sin^2\theta}} = d\theta$$

$= \theta + C$

$\theta = \sin^{-1}\left(\frac{x}{a}\right) - C$

now $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$ → wanna see?

Lets make $x = \frac{q}{\sqrt{LC}}$ $dx = \frac{dq}{\sqrt{LC}}$

so we integrate $\displaystyle\int \frac{\left(\frac{dq}{\sqrt{LC}}\right)}{\sqrt{\frac{2U}{L} - \left(\frac{q}{\sqrt{LC}}\right)^2}} = \int \frac{dt}{\sqrt{LC}}$

or, finally

$$\sin^{-1}\left[\frac{\left(\frac{q}{\sqrt{LC}}\right)}{\sqrt{\frac{2U}{L}}}\right] = \frac{t}{\sqrt{LC}} + \phi$$

take sin of both sides

$$\frac{\frac{q}{\sqrt{LC}}}{\sqrt{\frac{2U}{L}}} = \sin\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

or $q = \sqrt{LC}\sqrt{\frac{2U}{L}}\sin\left(\frac{t}{\sqrt{LC}} + \phi\right)$

$$q = \sqrt{2UC}\,\sin\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

if all the charge started on the capacitor $U = \frac{1}{2}\frac{q_0^2}{C}$

$$q = q_{max}\sin\left(\frac{t}{\sqrt{LC}} + \phi\right)$$

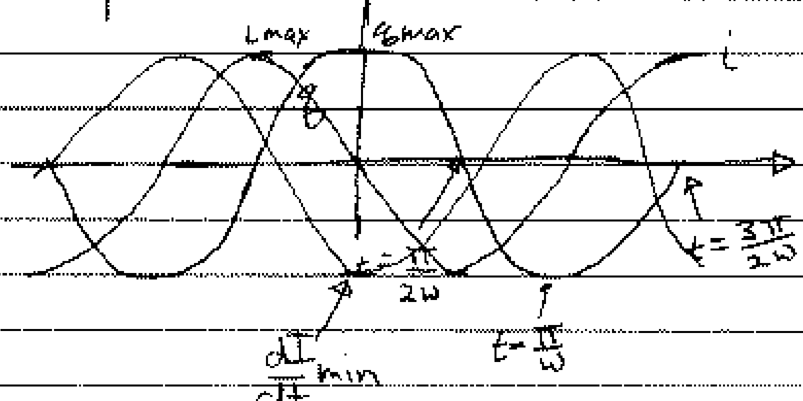
and at $t=0$ $q = q_{max}$ so $\phi = \frac{\pi}{2}$, $\omega = \frac{1}{\sqrt{LC}}$

$q = q_{max}\sin\left(\omega t + \frac{\pi}{2}\right)$

there are some really cool implications to this

look at current $\frac{dq}{dt} = \frac{q_0}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}} + \frac{\pi}{2}\right)$

lets plot them



voltage and current are out of phase i_{max} before q_{max}

Lets look at $\frac{dI}{dt} = \frac{d^2 q}{dt^2}$

$$= \frac{-q_0}{LC} \sin\left(\frac{t}{\sqrt{LC}} + \frac{\pi}{2}\right)$$

$\frac{dI}{dt}_{min}$

Now, how the heck do we keep this straight

note $V_{cap}^{max} = \frac{q_{max}}{C} = \mathcal{E}_C$

$V_{ind}^{max} = L \frac{dI}{dt}_{max} = \mathcal{E}_L$

and from our drawing

$\mathcal{E}_C$ peaks just before i_{max} $\quad i\ \mathcal{E}_C$

$\mathcal{E}_L$ peaks just after i_{max} $\quad \mathcal{E}_L\ i$

And we have a new friend to help us remember

$\mathcal{E}Li$ the $iC\mathcal{E}$ man

$\left(\begin{matrix}\mathcal{E} \text{ before } i \\ L\end{matrix}\right)$ $\quad$ $\left(\begin{matrix}i \text{ before } \mathcal{E} \\ C\end{matrix}\right)$

Why is this useful?

You see it all over the place

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

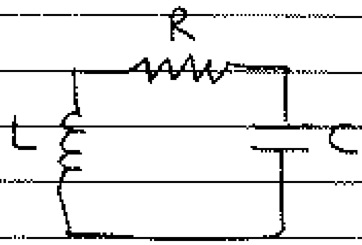
replace q with x and
L with m
C with $\frac{1}{k}$

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad ma + kx = 0$$

$U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

mass on spring!

in fact, some of the first "computers" were just capacitors inductors and resistors used to solve difficult mechanical problems now that we're comfortable with an "LC" circuit, lets add a resistor



~~For a discharging capacitor, in terms of the charge~~

$$\frac{q_c}{C} + L\frac{d\, i_c}{dt} + i_c R = 0$$

now this looks pretty daunting, lets look at this logically

- we know that there will be energy dissapated everytime current goes through the resistor

now, the amount of power in the resistor at any one time is

$$i_c^2 R$$

where $i_c = \frac{q_{max}}{\sqrt{LC}} \cos\left(\frac{t}{\sqrt{LC}} + \frac{\pi}{2}\right)$

Lets suppose R is really small so that we can assume the circuit oscillates at about the same frequency. So, over a whole oscillation period, the resistor will burn an amount of energy

$$\theta = \frac{t}{\sqrt{LC}} + \frac{\pi}{2}$$

$$\Delta U = \int I^2 R\, dt = \int_0^{2\pi} \frac{q^2_{max}}{LC} R \cos^2\theta \, d\theta \left(\sqrt{LC}\right)$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \pi$$

so $U(t) = u_0 e^{-R/L\, t} \sin^2(\omega t + \frac{\pi}{2}) = \frac{1}{2}\frac{q(t)^2}{C}$

expect $q(t) \sim q_0 e^{-\frac{R}{2L} t} \sin(\omega t + \frac{\pi}{2})$

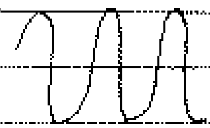
this is very close

the real answer is

$$q = q_0 e^{-\frac{R}{2L} t} \sin(\omega' t + \frac{\pi}{2})$$

$$\omega' = \sqrt{\omega^2 + (R/2L)^2} \qquad \omega^2 = \frac{1}{LC}$$

(adding resistance slows down the circuit)

instead of  we see 

and if we want to keep the oscillations going, we'll have to supply some power to the circuit from the outside

To do this, we're going to exploit the fact that

R has units of Ω

C has units of $\frac{s}{\Omega}$

L has units of Ωs

$$\Delta U = \frac{q^2_{max}}{\sqrt{LC}} R\pi$$

$$U_0 = \frac{q^2_{max}}{2C}$$

in one period $(U - \Delta U) = \frac{q^2_{max}}{2C}\left(1 - 2\pi \frac{RC}{\sqrt{LC}}\right) = \frac{q^2_{new}}{2C}$

so that in the next period

$$(U_{new} - \Delta U) = \frac{q^2_{new}}{2C}\left(1 - 2\pi\frac{RC}{\sqrt{LC}}\right) = \frac{q^2_{newer}}{2C}$$

or in N periods $= \frac{q^2_{max}}{2C}\left(1 - 2\pi\frac{RC}{\sqrt{LC}}\right)^N$

$$U_N = \frac{q^2_{max}}{2C}\left(1 - 2\pi\frac{RC}{\sqrt{LC}}\right)^N$$

now, you'll recognize that $e^{-\alpha t} \sim 1 - \alpha t$ if α is small

so, you can see, it's fairly easy to imagine, that our energy has an exponential fall off

$$e^{-\alpha\left(\frac{2\pi}{\omega}\right)} \approx 1 - \alpha\left(\frac{2\pi}{\omega}\right) = 1 - 2\pi\frac{RC}{\sqrt{LC}}$$

or $\alpha\left(\frac{2\pi}{\omega}\right) = \cancel{2\pi}\frac{RC}{\sqrt{LC}}$

$$\alpha = \frac{RC}{\sqrt{LC}}\omega \quad \left(\omega = \frac{1}{\sqrt{LC}}\right) \quad \alpha = \frac{RC}{LC} = \frac{R}{L}$$

so $U = U_0 e^{-\frac{R}{L}t}$

and the energy in the capacitor

$$U = U_0 \sin^2\left(\omega t + \frac{\pi}{2}\right)$$

now lets try an inductor

$$v = V\sin(\omega_0 t) \qquad V = L\frac{dI}{dt}$$

$$\frac{V}{L}dt = dI$$

$$I = \frac{V}{L}\int \sin\omega_0 t \, dt$$

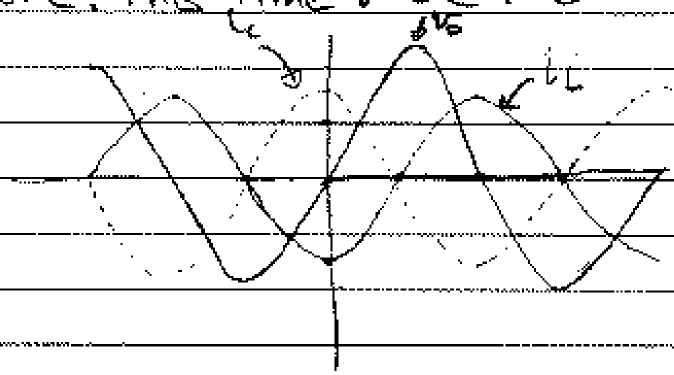
$$= -\frac{V}{\omega_0 L}\cos(\omega_0 t) \qquad \text{no constant}$$

$$= \frac{V}{\omega_0 L}\sin\left(\omega_0 t - \frac{\pi}{2}\right)$$

and the inductive reactance

$$X_L = \omega_0 L \qquad \frac{1}{s}\,\Omega s = \Omega$$

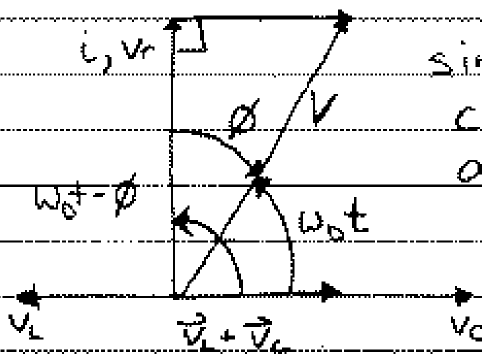
and all the maximum values are tracking like constant voltage values and we see the same behavior we saw before. This time $V=0$ @ $t=0$



like they're in parallel

ELI the ICE man again!

In order to find where the current is in relation to the driving voltage we add up all the vectors.



since we define the current to have an angle $\omega_0 t - \phi$ we have to define how we calculate it

here ϕ is negative

$$\text{and} \quad \tan\phi = \frac{|V_L| - |V_C|}{V_R} = \frac{iX_L - iX_C}{iR}$$

$$= \frac{X_L - X_C}{R}$$

and we know that V^2 is Σ of the sides of the triangle

$$V^2 = (V_L - V_C)^2 + V_r^2$$

$$= (iX_L - iX_C)^2 + (iR)^2 = i^2\left((X_L - X_C)^2 + R^2\right)$$

so $i_0 = V_0 / \sqrt{(X_L - X_C)^2 + R^2}$ max when $X_L = X_C$

$= V_0 / Z \leftarrow$ impedance $\qquad \omega_0 = \frac{1}{\sqrt{LC}}$

$i(t) = i_0 \sin(\omega_0 t - \phi) \qquad \phi = \frac{X_L - X_C}{R}$

now lets look at power

$$P = V_0 i_0 \sin(\omega_0 t) \sin(\omega_0 t - \phi)$$

use $\sin(\omega_0 t - \phi) = \sin\omega_0 t \cos\phi - \cos\omega_0 t \sin\phi$

$$P(t) = V_0 i_0 \left(\sin^2(\omega_0 t) \cos\phi - \underbrace{\sin(\omega_0 t)\cos(\omega_0 t)}_{\frac{1}{2}\sin\left(\frac{\omega_0 t}{2}\right)} \right)$$

whats the average power dissapated?

the average value of $\sin(\omega_0 t)\cos(\omega_0 t) = 0$

$\sin^2(\omega_0 t) = \frac{1}{2}$

$$avg = \frac{\int_0^{2\pi} \sin^2(\omega_0 t)\, d(\omega_0 t)}{\int_0^{2\pi} d(\omega_0 t)} = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$P_{avg} = \frac{1}{2} V_0 i_0 \cos\phi$$

$$= \frac{V_0}{\sqrt{2}} \frac{i_0}{\sqrt{2}} \cos\phi$$

what your meter reads

$$\cos\phi = \frac{V_r}{V} = \frac{iR}{iZ} = \frac{R}{Z}$$

$$= V_{rms}\, i_{rms} \cos\phi = V_{rms} I_{rms} \frac{R}{Z}$$

$$= V_{rms}^2 \frac{R}{Z^2}$$

max when Z smallest $X_L = X_C$

(burn max power in R for delivered I, V)

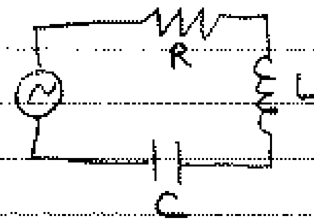
Now, lets say you have a lot of inductance in a circuit. For instance, you are running lot of electric motors

You are using $P_{av} = V_{rms} I_{rms} \cos\phi$

where $\cos\phi < 1$

Lets put all these circuit elements together in series

V sin($\omega_0 t$)



we want to find the current in terms of the voltage and reactances

$i = i_0 \sin(\omega_0 t - \phi)$

$V = V_0 \sin(\omega_0 t)$ convention!

In series, the same old rule apply

→ the current through all the ellements is the same

→ Σ voltages $= 0$ $\quad V\sin(\omega_0 t) = v_R(t) + v_L(t) + v_C(t)$

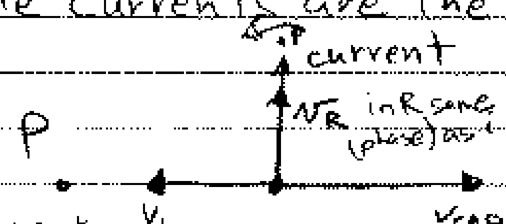
phase constant, ϕ, tells us the pos of i rel to v $\quad \phi_{neg}$ ϕ_{pos}

now since we know the current is the same, we could draw a graph

$X_L = X_C$

For this case, its easy $v_L + v_C$ cancel eachother

and try to add up the voltages at each point this is tough, what we do instead is treat each voltage as a vector since all the currents are the same



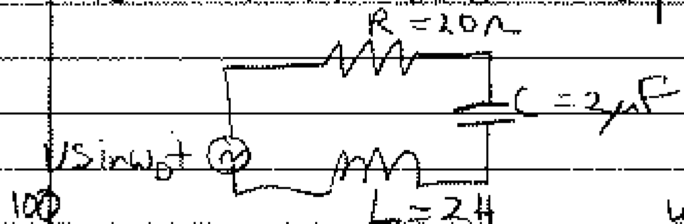
current passes a point before the voltage in a cap, V_{cap} @ $-90°$

Voltage will peak before current in a inductor

look at point P, as we rotate we see (E) I (CE) which is what we want.

great, so if we use alternating voltage, we can use transformers to make high voltage for long transmissions, and use trans. to step down this voltage so it is save for home use. well probably add a few capacitors in to get $\cos\phi = 1$ as well.

lets do a resonance problem



Are we at resonance?
what is Z
what is φ
what are V_R, V_L, V_C
what is resonance?

@ $\omega = 400\,\text{rad/s}$

resonance $= \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4\times10^{-6}}} = \frac{1}{2\times10^{-3}}$ = 500 rad/s

not there

$X_C = 1250$ (277.5 V!)
$X_L = 800$ (177.6 V)
$R = 20\Omega$ (4.4 V)

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}$$

$$= \sqrt{20^2 + \left(\frac{1}{400(2\times10^{-6})} - 400(2)\right)^2}$$

$X_L - X_C = 800 - 1250 = -450\Omega$

$\approx 450\Omega$

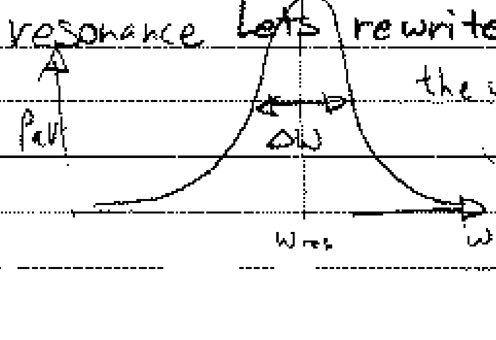
$\tan\phi = \frac{-450\Omega}{20\Omega} = -22.5$

$I_{max} = \frac{\varepsilon_{max}}{Z} = \frac{100}{450} = 0.222A$

$\phi = 87^\circ$
(not very eff)

$\frac{\text{Power we use}}{\text{Power bought}} = \cos\phi = .05!$

Power at resonance   Lets rewrite Z   Power $= \frac{V_{rms}^2 R}{Z^2}$



the width @ ½ max occurs when

$X_L - X_C = R$ $\omega L - \frac{1}{\omega C} = R$

$L^2(\omega^2 - \omega_0^2) = \omega^2 R$

$L(\omega + \omega_0)(\omega - \omega_0) = \omega R$

$\omega + \omega_0 \approx 2\omega$

$\omega - \omega_0 = \Delta\omega$

$L \frac{(\cancel{2}\omega)(\Delta\omega)}{\cancel{2}} \approx \omega R$

$\Delta\omega \sim \frac{R}{L}$

often hear about ~~quake~~ Q

define the quality of the peak as $\frac{\omega_0}{\Delta\omega}$ less width, sharper (lower R)

$Q = \frac{L\omega_0}{R}$ $\quad \omega_0 = \frac{1}{\sqrt{LC}}$

for this circuit

$Q = \frac{2(500)}{20} = 50$

$\Delta\omega = \frac{R}{L} = \frac{20}{2} = 10 \text{ rad/s}$

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

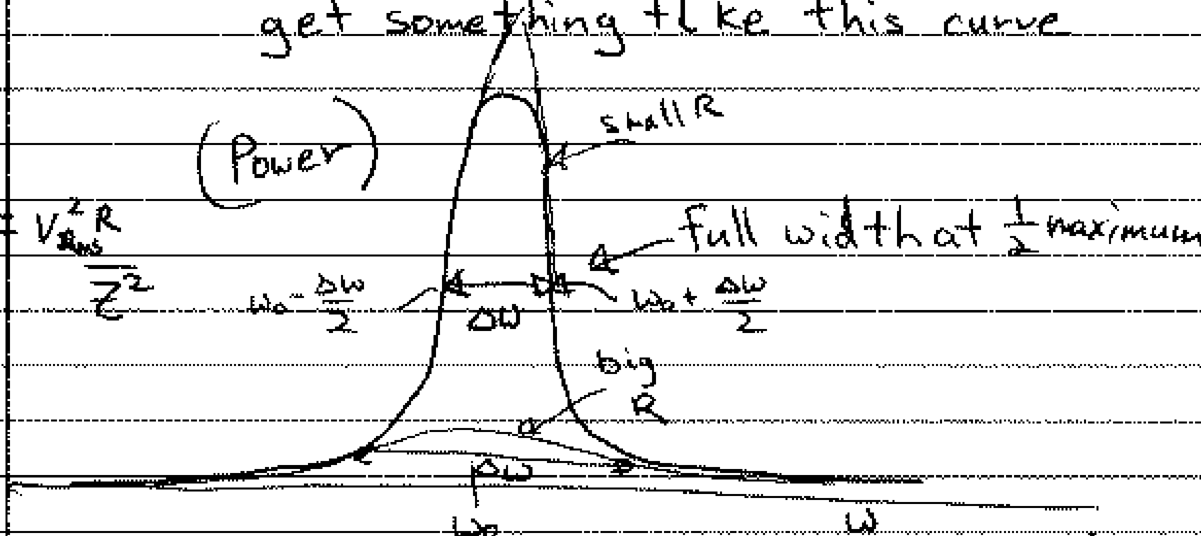
$$= \sqrt{R^2 + L^2\left(\omega - \frac{1}{\omega L C}\right)^2} \qquad \frac{1}{\sqrt{LC}} = \omega_0$$

$$= \sqrt{R^2 + L^2\left(\omega - \frac{\omega_0^2}{\omega}\right)^2}$$

$$= \sqrt{R^2 + \frac{L^2}{\omega^2}\left(\omega^2 - \omega_0^2\right)^2} \qquad \text{resonance at } \omega = \omega_0$$

get something like this curve

$$P = \frac{V_{RMS}^2 R}{Z^2}$$



resonance curves are usually described

by their sharpness or Q value $\frac{\omega_0}{\Delta\omega}$

the narrower the $\Delta\omega$, the sharper the resonance (the less R)

$\frac{1}{2}$ maximum power ocurrs when $Z \Rightarrow 2Z$

$\qquad\qquad R \qquad 2R$

or when $\frac{L^2}{\omega^2}(\omega^2 - \omega_0^2)^2 = R^2$

$$R = \frac{L}{\omega}(\omega^2 - \omega_0^2) = \frac{L}{\omega}(\omega + \omega_0)(\omega - \omega_0)$$

assuming a narrow peak $\omega \approx \omega_0$ $\quad R = \frac{L}{\omega_0}(\cancel{2}\omega_0)\left(\frac{\Delta\omega}{\cancel{2}}\right)$

big R low Q $\qquad \frac{\omega_0}{\Delta\omega} = \frac{L\omega_0}{R}$

Why is this useful? Spock would say:
(I suppose you guys say Mr Data)

Data: Captain,
To maximize the signal through the impedance of the circuit,
Please adjust the variable capacitor to
achieve resonance at our desired frequency.

Captain: You mean tune in the station on
the radio.

Data: I believe I said that.