Compressed States of Matter

We have already discussed White Dwarf Stars

Recall $\langle \varepsilon_f \rangle \propto \frac{1}{mass}$

and we said that the kinetic energy in the star will be dominated by the electrons

$$E_{tot} = \langle \varepsilon_f \rangle_{tot} + \text{Grav self energy}$$

$$= N_e \frac{3}{5} \frac{1}{2m_e} \left(\frac{h}{2}\right)^2 \left(\frac{3}{\pi} \frac{N_e}{\frac{4}{3}\pi R^3}\right)^{\frac{2}{3}} - \frac{3}{5} \frac{G M_{tot}^2}{R}$$

@ equilibrium $\quad \frac{dE_{tot}}{dR} = 0$

gave $\quad R \sim 7000$ km for the sun

or $\quad \langle \varepsilon_f \rangle = 122$ keV

Now, when $m_e + \langle \varepsilon_f \rangle$ gets to be about the difference in mass between the n & p or 1.3 MeV, the electrons have enough Q to make $e^- + p \to n + \nu_e$

So, eventually, the dominant KE comes from the neutrons. Since this accurs at a high energy for the electron, the electron energy $\langle \varepsilon_f \rangle$ goes more like pc

$$\langle \varepsilon_f \rangle = \langle pc \rangle = \int_0^{p_f} p^3 dp \Big/ \int_0^{p_f} p^2 dp$$

$$= \frac{3}{4} p_F c \;= \frac{3}{4}\left(\frac{hc}{2}\left(\frac{3}{\pi}\frac{N}{V}\right)^{1/3}\right)$$

$$E_{tot} = N_e \frac{3}{4} \frac{hc}{2} \left(\frac{3}{\pi} \frac{N_e}{\frac{4}{3}\pi R^3}\right)^{\frac{1}{3}} - \frac{3}{5} \frac{G M_{tot}^2}{R}$$

When $E_{tot} = 0$, the state is no longer bound (unstable)

Or when (assuming $pc \sim KE$)

$$N_e \frac{3}{4} \frac{hc}{2} \left(\frac{3}{\pi} \frac{N_e}{\frac{4}{3}\pi} \right)^{\frac{1}{3}} = \frac{3}{5} G N_e^2 m_n^2$$

$$\left(\frac{5}{3} \frac{1}{G m_n^2} \right) \left(\frac{3}{4} \frac{hc}{2} \right) \left(\frac{3}{\pi} \frac{1}{\frac{4}{3}\pi} \right)^{\frac{1}{3}} = N_e^{\frac{2}{3}}$$

or $N_e = \left[\frac{5}{3} \frac{1}{(6.67\times10^{-11} N \frac{m^2}{Kg^2})(1.67\times10^{-27} Kg)^2} \left(\frac{3}{4} \frac{(1240)(1.6\times10^{-19} J)(\times10^{-9} m)}{2} \right) \left(\frac{9}{4\pi^2} \right)^{1/3} \right]^{3}$

$= 8.2 \times 10^{57}$

This is actually about a factor of 4 higher than the correct limit computed by Chandrasekhar. Mostly I fudged. $KE = \sqrt{p^2c^2 + m^2c^4} - mc^2$ and you get some pretty nasty integrals.

The correct limit is $N_e = 1.71 \times 10^{57}$

or 1.44 m_{sun} in mass

(A crude correction @ $KE = 0.789$ MeV gives still $\times 2$ difference, refs. I found on the subject approach this from the viewpoint of pressure) eq. $PV = Nk_BT = \frac{2}{3} E$

$$P = \frac{2}{3} \frac{\langle E_f \rangle}{V} \text{ gas } \quad \& \quad \frac{GMm_n}{R^2} \frac{1}{4\pi R^2} \text{ on nucleon}$$

equilibrium is maintained as long as the pressures cancel or

$$\cancel{\frac{1}{V}} \frac{2}{4} \frac{hc}{2} \left(\frac{3}{\pi} \frac{N_e}{V} \right)^{1/3} = \frac{G N_e m_n^2}{3R} \cancel{\frac{1}{V}}$$

& in the above we'd replace $\frac{5}{3}$ with 2

$\Rightarrow$ worse!

corrections to the estimate:

1) we have said $N_e = N_H$ ← number of hadrons

realistically, there are some neutrons present.

In the sun, there are 3 ^1H for every ^4He

so, in reality there are ~ $\frac{5 \text{ electrons}}{7 \text{ nucleons}}$

$N_H = \left(\frac{7}{5} N_e\right)$

2) we have not done a good job of estimating the kinetic energy. When $Q = 0.789\,MeV = KE$

$E_{rel} = 1.3\,MeV$ & $pc = \sqrt{1.3^2 - 0.511^2} = 1.2\,MeV$

so, we need to correct our estimate by ~ $\frac{0.789}{1.2}$ roughly.

or $N_e^{new} = N_e^{old}\left(\left(\frac{5}{7}\right)^2\left(\frac{.789}{1.2}\right)\right)^{3/2}$

$= 0.194\, N_e^{old}$

$= 1.6 \times 10^{57}$

Which is much closer estimate, but we're still fudging a bit since we assume the fermi gas is degenerate. A finite temperature will increase p_f a bit. Trust me, the calculation gets pretty nasty. Our derivation gets the salient physics though.

⇒ at a certain point, the N_e will go down as the electrons combine with the protons. Recall for the non-relativistic case @ equilibrium $\frac{dE}{dR} = 0$ or

$\frac{N_e^{5/3}}{R^3} \propto \frac{N_H^2}{R^2}$ ← constant $\qquad R \propto N_e^{5/3}$

$$\mathcal{E}_f \propto \frac{N_e^{5/3}}{R^2} \propto \frac{1}{N_e^{5/3}}$$

so as N_e decreases, more slower electrons get more energy and combine with more protons until the electrons are gone and you end up with all neutrons.

This is a neutron star. The properties of this star are very similar to a white dwarf except we have m_n instead of m_e. And we expect

from book $\Rightarrow$ $$r_n = \frac{3^{4/3}\pi^{2/3}}{2^{4/3}} \frac{\hbar^2}{G m_n^3} N^{-1/3}$$

For $m_n = 1.5\ m_{solar}$

$N = 1.78 \times 10^{57}$

$r_n = 11\ km \qquad \rho = 4 \times 10^{14}\ g/cm^3$

nucleus $\dfrac{A\,(grams)}{(1.2\times10^{-13}\,cm)^3 A} \dfrac{1}{6.02\times10^{23}} = 9.6\times10^{14}$

$\Rightarrow$ pretty close to nuclear density

And we can guess that as the radius gets smaller, at a certain point, the pressures won't match. IF this happens when the neutrons are relativistic, we'd expect this to happen when $E_{tot} = 0$

$$\frac{N_H}{N_{sun}} = \frac{\left(2\cdot\frac{3}{5}\right)^{3/2}\ 8.2\times10^{57}}{1.2\times10^{57}} = 9 \text{ solar masses}$$

up to factor of 2 ish corrections.

we expect a black hole to occur when the radius of the compacted star is such that

$$\frac{1}{2} m v_{esc}^2 = \frac{G M_{star} m}{r_{star}} \quad \text{\& } v_{esc} = c$$

$$\text{or} \quad r_{star} = \frac{2 G M_{star}}{c^2} = \frac{2 G m_n N}{c^2}$$

& if we equate this to the radius of a neutron star

$$r_{star} = \frac{3^{4/3} \pi^{2/3}}{2^{4/3}} \frac{\hbar^2}{G m_n^3} N^{-1/3}$$ ← more is less

$$N = \left(\frac{3^{4/3} \pi^{2/3}}{2^{4/3}} \frac{\hbar^2 c^2}{(G m_n^2)^2} \right)^{3/4}$$

$$= \left(3.683 \cdot \frac{\left(1240\,\text{eVnm} \cdot 1.6\times10^{-19}\frac{\text{J}}{\text{eV}} \cdot \frac{1\times10^{-9}\text{m}}{\text{nm}} / 2\pi\right)^2}{\left(6.67\times10^{-11}\frac{\text{Nm}^2}{\text{kg}^2} (1.67\times10^{-27}\text{kg})^2\right)^2} \right)^{3/4}$$

$= 5.88 \times 10^{57}$ or ~ 5 solar masses

at which point $r_{star} = \left(3.683 \cdot \frac{\hbar^2 c^2}{(G m_n^2) m_n c^2}\right)(5.88\times10^{57})^{-1/3}$

$= 7242\,\text{m}$ (wow, that's tiny)

$$\text{\& } \langle \varepsilon_F \rangle = \frac{3}{5} \frac{1}{2 m_n c^2} \left(\frac{hc}{2}\right)^2 \left(\frac{3}{\pi} \frac{N}{\frac{4}{3}\pi R^3}\right)^{2/3}$$

≈ 300 MeV

not really very relativistic so our instinct about the limit is probably good, and we can check:

$\frac{2}{3} \varepsilon_F = 200\,\text{MeV}$, $\frac{G N_H m_n^2}{3R} = 314\,\text{MeV}$

& gravity wins