Review

Going to be talking about gaussian distribution today

- a limiting case of particle counting statistics
- usefulness in determining parameter & error propagation

Simple rules for particle decay

a) In a small interval of time, Δt, there is at most a single particle decay.

b) Probability for finding a decay is proportional to Δt.

c) Decays are independent

$$P_1(\Delta t) = M \Delta t \qquad (M = \text{constant})$$

probability of finding 1 decay in Δt

so $$P_0(\Delta t) = 1 - M\Delta t$$

Now we employ a trick

Probability of getting N decays in time $t + \Delta t$

= Probability of getting N decays in t & none in Δt

+ Probability of getting N-1 in t & one in Δt

$$P_N(t+\Delta t) = P_N(t)\, P_0(\Delta t) + P_{N-1}(t)\, P_1(\Delta t)$$

$$= P_N(t)(1 - M\Delta t) + P_{N-1}(t)\, M \Delta t$$

$$\frac{P_N(t+\Delta t) - P_N(t)}{\Delta t} = -M P_N(t) + M P_{N-1}(t)$$

taking Δt small

$$\frac{dP_N(t)}{dt} = -M P_N(t) + M P_{N-1}(t)$$

has solution

$$P_N(t) = \frac{1}{N!}(Mt)^N e^{-Mt}$$

where $$\sum_{N=0}^{\infty} \frac{1}{N!}(Mt)^N e^{-Mt} = 1$$

(where $N!$ comes from)

$$\boxed{\hookrightarrow \sum_{N=0}^{\infty} \frac{(Mt)^N}{N!} = e^{Mt}}$$

Called Poisson distribution

(Properties)

consider the most likely value in a distribution (In Q.M. this was an expectation value $\frac{\int \psi_x^* \psi \, dx}{\int \psi^* \psi \, dx} = \langle x \rangle$)

$$\langle x \rangle = \frac{\sum_a^b x f(x)}{\sum_a^b f(x)} \Rightarrow \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}$$

discrete

example: Flat distribution between 1 & 2

$$\langle x \rangle = \frac{\int_1^2 x\, dx}{\int_1^2 dx} = \frac{\frac{1}{2}x^2\Big|_1^2}{x\Big|_1^2} = \frac{\frac{3}{2}}{1} = \frac{3}{2}$$

right in the middle

Consider the following dirty trick:

$$\sum_{N=0}^{\infty} \frac{a^N}{N!}(Mt)^N e^{-Mt} = \sum_{N=0}^{\infty} \frac{1}{N!}(aMt)^N e^{-aMt}\left(e^{(aMt-Mt)}\right)$$

$$= e^{(aMt-Mt)} \quad \text{i.e., its } \langle e^{(aMt-Mt)} \rangle$$

and

$$\frac{d}{da}\sum_{N=0}^{\infty} \frac{a^N}{N!}(Mt)^N e^{-Mt} = \sum_{N=0}^{\infty} \frac{Na^{N-1}}{N!}(Mt)^N e^{-Mt}$$

$$= \sum_{N=0}^{\infty} \frac{N}{a}\frac{a^N}{N!}(Mt)^N e^{-aMt}\left(e^{(aMt-Mt)}\right)$$

$$= \left\langle \frac{N}{a} e^{aMt-Mt} \right\rangle$$

$$= \langle N \rangle \frac{e^{aMt-Mt}}{a} = \langle N \rangle \text{ as } a \Rightarrow 1$$

$$= \frac{d}{da}\left(e^{aMt-Mt}\right) = Mt\left(e^{aMt-Mt}\right)$$

$$= Mt \quad \text{as } a \Rightarrow 1$$

$$\text{or} \quad \langle N \rangle = Mt$$

Now, as we saw from the plot of the Poisson distribution, there is a spread of values associated with a particular value of Mt.

Lets look at a distribution that closely resembles the Poisson distribution. It's called the Gaussian distribution, and it has the following form

$$P_{(x)} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left((x-\mu)/\sigma\right)^2} \quad \left(\text{notice we replaced } N \text{ with } x\right)$$

where μ is the mean and σ is characteristic of the spread of values and is called σ.

This is the familiar bell curve. (ξ $\mu = Mt$ from above)

The value σ comes from the expectation value of $(x-\mu)^2$

let $a = \frac{1}{2\sigma^2}$ & $y = (x-\mu)$

$$\langle (x-\mu)^2 \rangle = \int_{-\infty}^{\infty} y^2 e^{-ay^2} dy \Big/ \int_{-\infty}^{\infty} e^{-ay^2} dy$$

$$= \left(-\frac{d}{da} \int_{-\infty}^{\infty} e^{-ay^2} dy\right) \Big/ \int_{-\infty}^{\infty} e^{-ay^2} dy$$

note

$$\boxed{\begin{gathered} \left(\int_{-\infty}^{\infty} e^{-ay^2} dy = \sqrt{\frac{\pi}{a}} \right) \\ \text{comes from} \\ \sqrt{\int_{-\infty}^{\infty} e^{-ax^2} dx \int_{-\infty}^{\infty} e^{-ay^2} dy} \\ = \sqrt{\int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r\,dr\,d\theta} \\ u = r^2 \quad du = 2r\,dr \\ \rightarrow \sqrt{2\pi \int_0^{\infty} e^{-au} \frac{du}{2}} = \sqrt{\frac{\pi}{a}} \end{gathered}}$$

$$= \left(-\frac{d}{da} \sqrt{\frac{\pi}{a}}\right) \Big/ \sqrt{\frac{\pi}{a}}$$

$$= \frac{1}{2} \frac{1}{a} \sqrt{\frac{\pi}{a}} \Big/ \sqrt{\frac{\pi}{a}}$$

$$= \sigma^2$$

For numbers distributed via a gaussian distribution 68% of the data lies between $\pm\sigma$ of μ. $\Rightarrow$ spread $= \sqrt{\sigma^2}$

& for our discrete distribution

$$\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 - 2N\langle N \rangle + \langle N \rangle^2 \rangle$$

$$= \langle N^2 \rangle - 2\langle N \rangle \langle N \rangle + \langle N \rangle^2 = \langle N^2 \rangle - \langle N \rangle^2$$

$$\langle N^2 \rangle = \left(\frac{d^2}{da^2} + \frac{d}{da}\right) e^{aMt - Mt}\Big|_{a=1} = (Mt)^2 + Mt$$

$$\frac{d^2}{da^2} \sum_{N=0}^{\infty} \frac{a^N}{N!} (Mt)^N e^{-Mt} = \frac{d}{da} \sum_{N=1}^{\infty} \frac{N a^{N-1}}{N!} (Mt)^N e^{-Mt}$$

$$= \sum_{N=1}^{\infty} \frac{(N^2 - N)}{a^2} \frac{a^N}{N!} (Mt)^N e^{-Mt}$$

$$\text{spread} = \sqrt{(Mt)^2 + Mt - (Mt)^2} = \sqrt{(Mt)}$$
$$= \sqrt{N}$$

spread of sum of several N's

$$N_{tot} = N_1 + N_2 + N_3 + \dots$$
$$\text{spread}\,(N_{tot}) = \sqrt{N_{tot}} = \sqrt{N_1 + N_2 + N_3 + \dots}$$
$$= \sqrt{(\text{spread}(N_1))^2 + (\text{spread}(N_2))^2 + (\text{spread}(N_3))^2 + \dots}$$

or, in general

$$\sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \dots$$

suppose you have a function that depends on N

$$f(N) = 6N$$

we expect the spread on $f(N)$ to be $6\sqrt{N}$

or in general

$$d\,f(N) = \frac{\partial f(N)}{\partial N} dN$$

$$\Delta f(N) = \underbrace{\frac{\partial f(N)}{\partial N}}_{6} \underbrace{\Delta N}_{\sqrt{N},\ \text{or}\ \sigma} \qquad \frac{\partial f(N)}{\partial N} = 6$$

& for more than one variable, that has its own σ

$$dV = \underbrace{\frac{\partial f(x_1, x_2, x_3 \dots)}{\partial x_1} dx_1}_{} + \underbrace{\frac{\partial f(x_1, x_2, x_3 \dots)}{\partial x_2} dx_2}_{} + \dots$$

if x_1 & x_2 are independent — these add like σ's above

$$\sigma_V \text{ or } \Delta V = \sqrt{\left(\frac{\partial f}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\sigma_{x_2}\right)^2 + \left(\frac{\partial f}{\partial x_3}\sigma_{x_3}\right)^2 + \dots}$$

suppose that we measure Mt many times and we know what M & t are. We would expect to see N distributed via the Poisson distribution, & we should be able to calculate an average for $\langle N \rangle$

$$\langle N \rangle = \mu = \frac{\sum Mt \text{ measurements}}{\# \text{ of trials}}$$

so

$$\#\text{trials}(\mu) = \sum N_i$$

better way to do this on pgs 9 & 10 $\Rightarrow$

$$\Delta(\#\text{trials}(\mu)) = \sqrt{\sum N_i}$$

$$\Delta\mu = \frac{\sqrt{\sum N_i}}{\#\text{trials}} = \frac{\sqrt{\#\text{trials}}\,\sqrt{\mu}}{\#\text{ trials}}$$

$$= \sqrt{\mu}/\sqrt{\#\text{trials}}$$

averaging helps

Actually, you don't even need a Gaussian to get a Gaussian distribution. In a random process, the sum of the outcomes tends toward a gaussian distribution.

ex Random number uniformly distributed between 0 & 1 expect $\langle (x-\mu)^2 \rangle = \dfrac{\int_0^1 (x-0.5)^2 dx}{\int_0^1 dx}$

$$= \frac{(x-0.5)^3}{3} = \frac{1}{12}, \quad \sigma = \frac{1}{\sqrt{12}}$$

so, if we add 4 of these together, we expect a distribution with $\sigma_{tot} = \sqrt{\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}} = \frac{1}{\sqrt{3}}$

We can exploit this behavior.

Suppose we have a set of points that are distributed according to a gaussian distribution. We'd expect each point to be a little different from the average we started with, but if we looked at a bunch of points, we expect to be able to make a pretty good estimate for σ:

$$\sigma^2 = \sum_{points} (x_i - \mu)^2 \Big/ \#\text{ of points}$$

suppose you didn't know μ? You'd compute it

$$\mu = \frac{\sum_{points} x_i}{\#\text{ points}}$$

but σ^2 now becomes

$$\sigma^2 = \sum_{points} (x_i - \mu)^2 \Big/ (\#\text{ points} - 1)$$

This makes sense if you think about it

1) for one point, $\sigma = 0$, can't estimate it from the data

2) for many points, suppose we gave each its own μ $\quad \sigma^2 = \sum_{point} (x_i - \mu_i)^2 \Big/ (\#\text{ points} - ?)$

σ^2 would be 0 if we found the average for each point separately.

In general, each time you use the data to find a parameter, you need to reduce the number of points by one.

$$\sigma^2 = \sum_{\text{points}} (x - f(x))^2 / \text{degrees of freedom}$$

where degrees of freedom = # points − # parameters you need if f(x)

Indeed, it is very useful to turn this approach back on itself to determine parameters. This is especially useful if one can estimate the errors on the points before hand.

ex average

suppose you took several trials of counting the number of decays from a radio active source

1) we'll assume you count all the decays
2) we'll assume there is only one process for decay

For each measurement of the number of counts in a time interval, you associate an error of $\sqrt{N}$, this is cheating a little, but its not too bad on average, so you expect

$$\#\text{ degress of Freedom} = \sum_{i=1}^{\#\text{ measurements}} \frac{(N_i - \mu)^2}{(\sqrt{N_i})^2} = \chi^2$$

called chi-squared.

If you are trying to find μ, the best μ should give you the minimum χ^2.
So for estimating parameters, you are trying to find values that minimize χ^2. If you've done a good job $\chi^2 \sim$ # degrees of freedom

ex Average

$$\chi^2 = \sum_{points} \frac{(N_i - \mu)^2}{(\sqrt{N_i})^2}$$

$$\frac{\partial \chi^2}{\partial \mu} = 0 \text{ at minimum (or extrema)}$$

$$= \sum_{i=1}^{points} 2 \frac{(N_i - \mu)}{(\sqrt{N_i})^2} \qquad \left(\text{in general } \sum_{i=1}^{\# points} \frac{(x_i - \mu)^2}{\sigma_i^2} \right)$$

$$\sum_{i=1}^{\# points} \frac{N_i}{(\sqrt{N_i})^2} = \sum_{i=1}^{\# points} \frac{\mu}{(\sqrt{N_i})^2}$$

$$\frac{\sum_{i=1}^{\# points} \frac{N_i}{(\sqrt{N_i})^2}}{\sum_{i=1}^{points} \frac{1}{(\sqrt{N_i})^2}} = \mu = \frac{\# points}{\sum_{i=1}^{\# points} \frac{1}{N_i}} \qquad \text{in general} \left(\frac{\sum \frac{x_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}} \right)$$

a bit different, but really cool. This gives you a mechanism for down weighting points with large errors.
Suppose you measure the following:

$$\left.\begin{array}{l} 0.6 \pm 0.1 \\ 0.5 \pm 0.2 \\ 0.7 \pm 0.2 \\ 1.5 \pm 1.0 \end{array}\right\} \text{straight average} = 8.5 \qquad \text{weighted average} = 0.606$$

What is the error on this average?

Consider the general case

$$\mu = \frac{\sum_i^{\#} \frac{x_i}{\sigma_i^2}}{\sum_j^{\#} \frac{1}{\sigma_j^2}}$$

$$\frac{\partial \mu}{\partial x_i} = \frac{\frac{1}{\sigma_i^2}}{\sum_j^{\#} \frac{1}{\sigma_j^2}}$$

$$d\mu \text{ or } \Delta\mu = \sqrt{\sum_i^{\#} \left(\frac{\frac{1}{\sigma_i^2}}{\sum_j^{\#} \frac{1}{\sigma_j^2}} \right)^2 \sigma_i^2}$$

$$= \sqrt{\frac{\sum_i^{\#} \frac{1}{\sigma_i^2}}{\left(\sum_j^{\#} \frac{1}{\sigma_j^2} \right)^2}}$$

$$= \sqrt{1 \Big/ \sum_j^{\#} \frac{1}{\sigma_j^2}}$$

in our example

Weighted average = 0.606

error = 0.08

answer 0.61 ± 0.08