Consider using this to find out what the average energy is of a bunch of oscillators at temperature T

we know $N_{i+1} = N_i e^{-(\mathcal{E}_{i+1} - \mathcal{E}_i)/k_BT}$ for each energy level
let $\mathcal{E}_0 = 0$

This is still a sum of quanta, with one frequency, ω
$$\langle E \rangle = \frac{E_{Tot}}{N_{tot}} = \frac{\hbar\omega\left(0 + N_0 e^{-\hbar\omega/k_BT} + 2\left(N_0 e^{-\hbar\omega/k_BT}\right)e^{-\hbar\omega/k_BT} + 3(\ldots\right)}{N_0 + \left(N_0 e^{-\hbar\omega/k_BT}\right) + \left(N_0 e^{-\hbar\omega/k_BT}\right)e^{-\hbar\omega/k_BT} + \ldots}$$

note $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 +$$

substitute $$\frac{e^{-\hbar\omega/k_BT}}{\left(1-e^{-\hbar\omega/k_BT}\right)^2} = \left(0 + e^{-\hbar\omega/k_BT} + 2\left(e^{-\hbar\omega/k_BT}\right)e^{-\hbar\omega/k_BT} \ldots\right.$$

$$\frac{1}{\left(1-e^{-\hbar\omega/k_BT}\right)} = 1 + e^{-\hbar\omega/k_BT} + \left(e^{-\hbar\omega/k_BT}\right)e^{-\hbar\omega/k_BT}$$

$$\Rightarrow \quad \langle E \rangle = \frac{\hbar\omega N_0\left(\frac{e^{-\hbar\omega/k_BT}}{\left(1-e^{-\hbar\omega/k_BT}\right)^2}\right)}{N_0\left(\frac{1}{1-e^{-\hbar\omega/k_BT}}\right)}$$

$$= \hbar\omega \frac{e^{-\hbar\omega/k_BT}}{1-e^{-\hbar\omega/k_BT}} = \hbar\omega \frac{1}{e^{\hbar\omega/k_BT}-1}$$

For a collection of 3N oscillators

$$E_{tot} = 3N \frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}$$

$$\frac{dE}{dT} = C_v = 3N\hbar\omega \frac{-1}{\left(e^{\hbar\omega/k_BT}-1\right)^2}\left(-\frac{\hbar\omega}{k_BT^2}\right)e^{\hbar\omega/k_BT}$$

(See page 323 of your book)

$$= 3Nk_B \, e^{\hbar\omega/k_BT} \frac{1}{\left(e^{\hbar\omega/k_BT}-1\right)^2}\left(\frac{\hbar\omega}{k_BT}\right)^2$$

when T gets large

$$\sim 3Nk_B\left(1 + \frac{\hbar\omega}{k_BT}\right)\frac{1}{\left(\frac{\hbar\omega}{k_BT}\right)^2}\left(\frac{\hbar\omega}{k_BT}\right)^2$$

$\sim 3Nk_B$ cool

Actually, the way Einstein 1st did this problem was to assume the solid had one quantized mode of vibration (quanta is called a phonon) and that the energy in a solid was stored by phonons of a particular frequency.

$$\langle E \rangle = (\text{Energy of a phonon}) \cdot (\#\text{ of phonons})$$

average energy of a phonon

$$\underset{\text{(adjustable parameter)}}{(\hbar\omega)} \left(\frac{1}{e^{\hbar\omega/k_BT}-1}\right)$$

We can turn this around and say what kind of a contribution to the total energy does one particular frequency of vibration make.

In a cavity at temerature T with a small hole, we know that there is a particular distribution of frequencies. We should be able to figure out what that is. We'll treat each frequency as a collection of oscillators. Recall, the conundrum, according to classical theory you should be flooded with x rays from a black body.

<u>Em waves in a box</u> must vanish at walls

$$\lambda_x = \frac{2L}{n_x} \quad \text{or} \quad \omega_x = \frac{\pi c n_x}{L}$$

getting multiplicity { wave will be described by n_x, n_y, n_z & 2 polarizations; to count up states, treat n as r

$n^2 = n_x^2 + n_y^2 + n_z^2$	$r^2 = x^2 + y^2 + z^2$
# of states at n, dn	Volume of shell at r, dr
$\left(2 \cdot \frac{1}{8}\right) 4\pi n^2 dn$	$4\pi r^2 dr$

polarizations

positive integers

so # of frequencies at $\omega, d\omega$

$$\pi\left(\frac{L}{\pi c}\right)^3 \omega^2 d\omega$$

$$\#\text{ photons at a particular frequency} = \pi\left(\frac{L}{\pi c}\right)^3 \omega^2 \left(\frac{1}{e^{\hbar\omega/k_BT}-1}\right)$$

So now we can calculate the energy contribution at each frequency

$$E(\omega)\,d(\omega) = \underbrace{\pi\left(\frac{L}{\pi c}\right)^3 \omega^2 (\hbar\omega)}_{\frac{\text{energy}}{\text{unit volume}}\frac{1}{(\text{frequency range})}} \left(\frac{1}{e^{\hbar\omega/k_BT}-1}\right) d\omega$$

The distribution of $\frac{E(\omega)}{V}$ becomes $\frac{\hbar}{\pi^2 c^3}\,\frac{\omega^3}{e^{\hbar\omega/k_BT}-1}$

which is Planks Black body formula
the total $\frac{E}{V}$ coming from the hole is $\propto T^4$.
which you can get from integrating the above over all frequencies.

(Incidentally, this is also how our solid behaves at low temeratures. Einstein was almost right, but at lower temperature, the solid acts more like a black body, having modes involving the whole solid. The idea is that the solid has a limited number of modes with which it can oscillate though. Eventually being limited by the number of atoms present.

→ There are 3 polarizations for elastic waves in a solid

so $\quad 3\cdot\frac{1}{8}4\pi\int_0^{n_0} n^2\,dn = 3N$

$$\sum_{\text{states}}\begin{pmatrix}\text{density}\\\text{of}\\\text{states}\end{pmatrix}\left(\begin{matrix}\#\text{ quanta}\\\text{in state}\end{matrix}\right)(\text{Energy}) \Rightarrow \frac{3\pi}{2}\int_0^{n_0}\left(\frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}\right)n^2\,dn$$

at low temperature $\hbar\omega/k_BT$ can be large

$x = \frac{\hbar\pi c n}{L k_B T}$ $\quad dx = \frac{\hbar\pi c}{L k_B T}\,dn$ $\quad$ & change $\int_0^{n_0}$ to $\int_0^{x_B} \sim \int_0^{\infty}$ at low T

& we get back the blackbody result