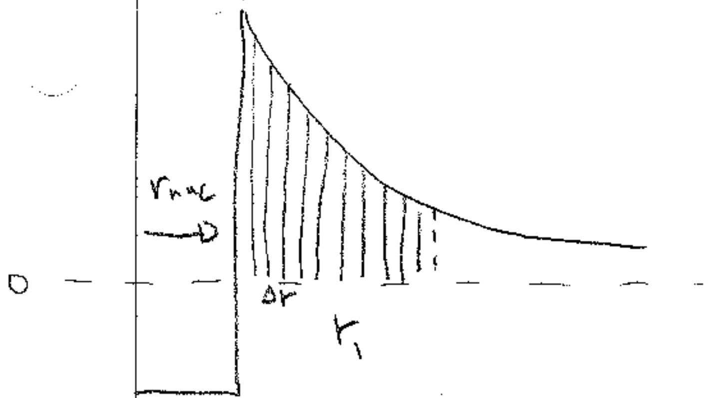
If we think of the coulomb barrier as made up of slices



The Transmission probability will be a product of all the individual slabs

$$T = \prod_i e^{-2\alpha(r_i)\Delta r_i}$$

$\prod_i$: product; $\alpha(r_i)$: α as a function of r_i; Δr_i: size of slice

Why is T small?
IF $KE \sim 6$ MeV
For an α particle, it is moving at 10% c in the nucleus. For $A=209$, this means in 1s, the α hits the barrier

$$\frac{0.1\times 3.0\times 10^{23}\frac{fm}{s}\cdot 1s}{7.12\,fm\cdot 2}$$

About 10^{21} times!
$t_{1/2}$ goes from 10^{-6}s to 10^{18}s

We can re-write this as $\qquad r_1 = \frac{k2(Z-2)e^2}{E}$

$$T = e^{-2\int_{r_{nuc}}^{r_1} \frac{\sqrt{2m}}{\hbar}\sqrt{E}\left(\sqrt{\frac{r_1}{r}-1}\right)dr}$$

let $r = r_1\sin^2\theta \quad dr = r_1\, 2\sin\theta\cos\theta\, d\theta$

$$\ln T = -4\int_{\sin^{-1}\sqrt{\frac{r_{nuc}}{r_1}}}^{\pi/2}\sqrt{\frac{2mE}{\hbar^2}}\, r_1\cos^2\theta\, d\theta$$

$$= -4\sqrt{\frac{2mE}{\hbar^2}}\, r_1\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Bigg|_{\sin^{-1}\sqrt{\frac{r_{nuc}}{r_1}}}^{\pi/2}$$

$$\left\{\begin{array}{l} r_{nuc}/r_1 \sim 7\times10^{-15}m\cdot\dfrac{6\times10^{6}eV\times 1.6\times10^{-19}J/eV}{9\times10^{9}\frac{Nm^2}{C^2}\cdot 2(80)(1.6\times10^{-19}C)^2} = 0.18 \\ \sin^{-1}\sqrt{\dfrac{r_{nuc}}{r_1}} \sim \sqrt{\dfrac{r_{nuc}}{r_1}} \end{array}\right\}$$

$$\ln T = -4\sqrt{\frac{2mE}{\hbar^2}}\, r_1\left[\frac{\pi}{4} - \frac{1}{2}\sqrt{\frac{r_{nuc}}{r_1}} - \frac{1}{4}(2)\sqrt{\frac{r_{nuc}}{r_1}}\right]$$

$$= 4\sqrt{\frac{2mE}{\hbar^2}}\, r_1\left[\sqrt{\frac{r_{nuc}}{r_1}} - \frac{\pi}{4}\right]$$

now, put back in $r_1 = \frac{2kZ'e^2}{E}$

$$\ln T = 4\sqrt{\frac{2mE}{\hbar^2}}\,\frac{2kZ'e^2}{E}\left[\sqrt{\frac{E}{2kZ'e^2}}\sqrt{r_{nuc}} - \frac{\pi}{4}\right]$$

$$= \frac{8}{\hbar}\sqrt{mkZ'e^2 r_{nuc}} - \frac{2\pi kZ'e^2}{\hbar}\sqrt{\frac{2m}{E}}$$

The activity of a large sample is going to be

$$N \times \frac{\text{Prob to tunnel}}{s} = N\lambda$$

$$\text{or} \quad \underbrace{\left(\frac{\text{velocity}}{2R}\right) T}_{\frac{\text{Prob to tunnel}}{s}} = \lambda$$

$$t_{1/2} = \frac{\ln(2)}{\lambda} = \ln(2)\left(\frac{2r_{nuc}}{v}\right)\frac{1}{T}$$

$$\ln(t_{1/2}) = \ln\left(\ln(2)\frac{2r_{nuc}}{\sqrt{2E/m}}\right) + \frac{2\pi k z' e^2}{\hbar}\sqrt{\frac{2m}{E}} - \frac{8}{\hbar}\sqrt{kz'e^2 r_{nuc} m}$$

2 examples

$^{226}_{88}Ra \qquad E = 4.06\,MeV \qquad t_{1/2} = 1.62\times10^{3}\,y$

$^{212}_{84}Po \qquad E = 8.95\,MeV \qquad t_{1/2} = 3.0\times10^{-7}\,s$

$m_\alpha = 3.728\,GeV/c^2$

calculate

expect $\qquad$ get

$\ln(1.62\times10^{3}_{y})$ $\quad \ln\left(\frac{\ln(2)\;2(1.2\,fm)226^{1/3}}{\sqrt{2(4.06/3728}\;(3.0\times10^{23}\,fm/s)}\right) \Rightarrow (-48.68)$

$$+\frac{(2\pi)^2(9\times10^9\,\frac{Nm}{c^2})(86)(1.6\times10^{-19}C)^2}{hc \rightarrow (1240\,eV\times10^{-9}m)\cdot \cancel{1.6\times10^{-19}J/eV}}\sqrt{\frac{2(3728\,MeV)}{4.06\,MeV}} \Rightarrow (168.96)$$

$$-\frac{8(2\pi)}{\underbrace{1240\,eV\times10^{-9}m}_{hc}}c\sqrt{3728\times10^{6}\frac{eV}{c^2}}\sqrt{9\times10^{9}\frac{Nm^2}{c^2}\cdot(86)\frac{(1.6\times10^{-19}C)^2}{1.6\times10^{-19}J/eV}\cdot 1.2\times10^{-15}m\,(226)^{1/3}} \Rightarrow (-74.44)$$

expect

$\ln\left(1.62\times10^{3}\,y\cdot\frac{365d}{y}\cdot\frac{24h}{d}\cdot\frac{3600s}{h}\right)$ $\quad$ get $\quad -48.96 + 168.96 - 74.44$

$= 24.66 \qquad\qquad\qquad 45.56$

expect $\ln(3.0\times10^{-7}s)$ get $\ln\left(e^{-48.68}\left(\frac{212}{226}\right)^{1/3}\cdot\sqrt{\frac{4.06}{8.95}}\right.$

$$+168.96\cdot\left(\frac{84}{86}\right)\sqrt{\frac{4.06}{8.95}}-74.44\cdot\sqrt{\frac{84}{86}}\left(\frac{212}{226}\right)^{\frac{1}{6}}$$

expect -15 get $-49.09+110.52-72.79$

-11.36

not too bad

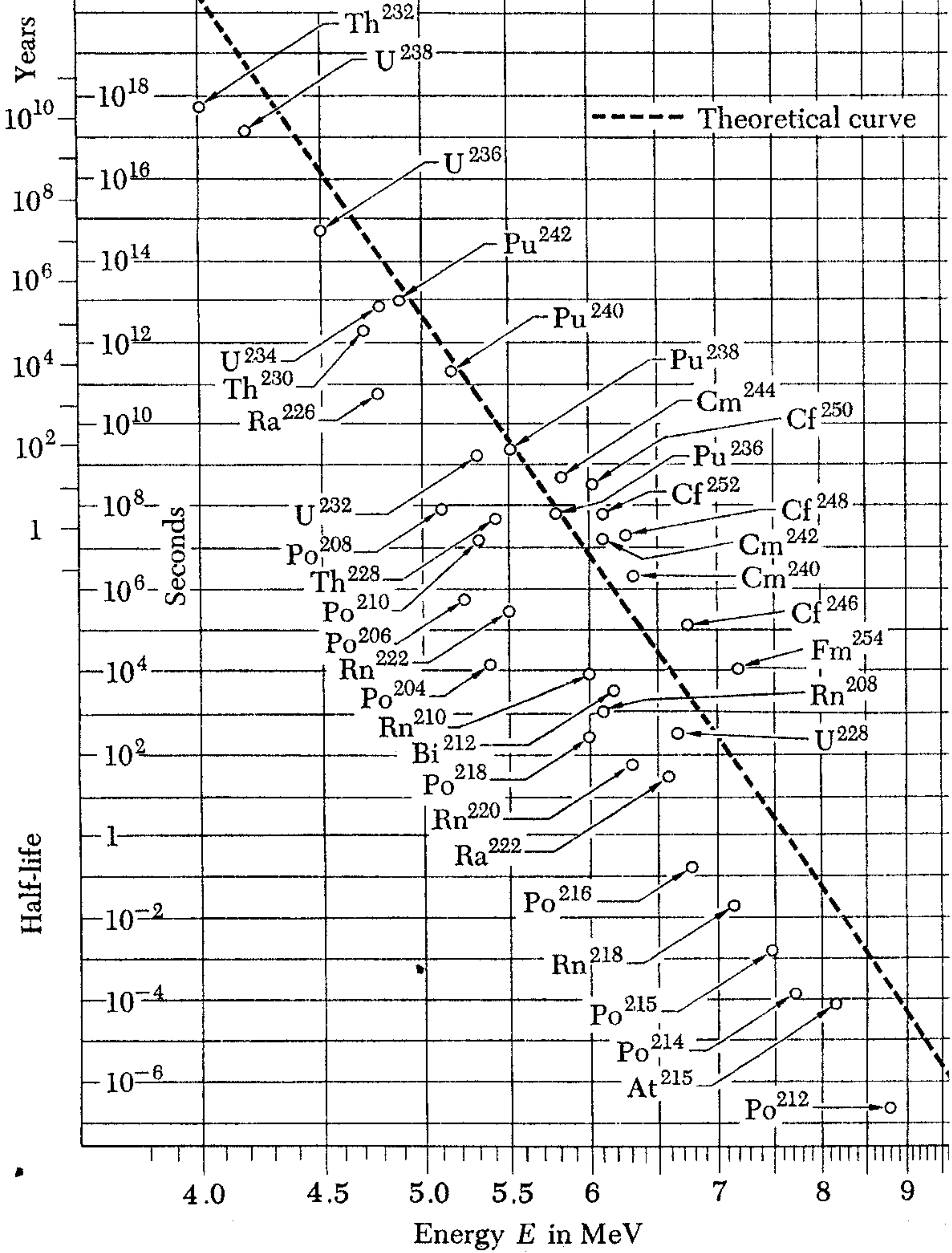
notice that it's the 2nd & 3rd terms that really changed

for these large nuclei $A\sim 3Z$ & you can make the relation

$$\log_{10} t_{1/2} = 1.61\left(Z/\sqrt{E} - Z^{2/3}\right) - 28.9$$

which your more likely to find in the literature (might be called Geiger-Nuttall rule)

$\left\{\text{The numbers for the equation above come from a calculation done}\right\}$


Years
Half-life
Seconds
Theoretical curve
10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
1
10^{18}
10^{16}
10^{14}
10^{12}
10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
1
10^{-2}
10^{-4}
10^{-6}
Th^{232}
U^{238}
U^{236}
Pu^{242}
Pu^{240}
U^{234}
Th^{230}
Ra^{226}
Pu^{238}
Cm^{244}
Cf^{250}
Pu^{236}
Cf^{252}
Cf^{248}
U^{232}
Po^{208}
Th^{228}
Cm^{242}
Cm^{240}
Po^{210}
Po^{206}
Cf^{246}
Rn^{222}
Fm^{254}
Po^{204}
Rn^{210}
Rn^{208}
Bi^{212}
U^{228}
Po^{218}
Rn^{220}
Ra^{222}
Po^{216}
Rn^{218}
Po^{215}
Po^{214}
At^{215}
Po^{212}
4.0
4.5
5.0
5.5
6
7
8
9
Energy E in MeV