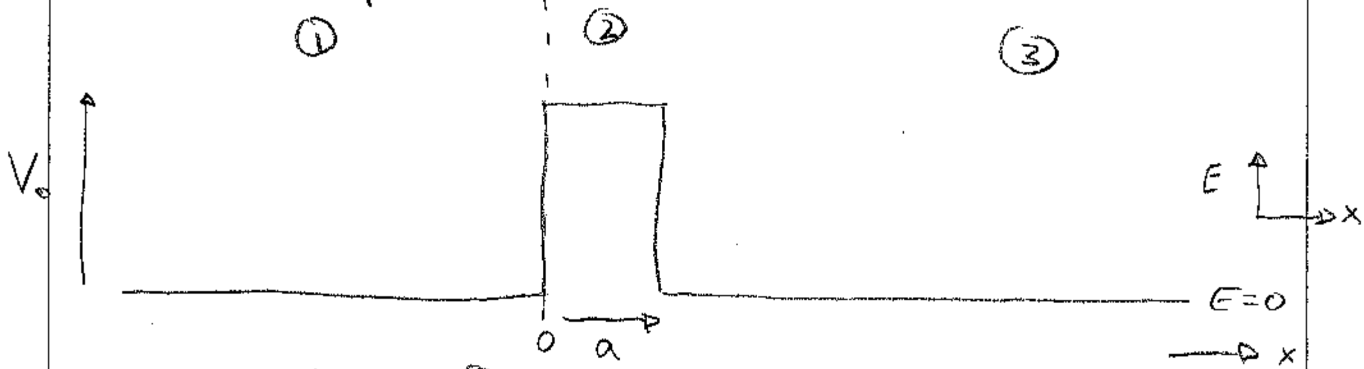
1st, we need to solve the classic "barrier Penetration" problem



In Region 1 the wave function has solutions like $Ae^{ik_1x} + Be^{-ik_1x}$

In 2 $\quad Ce^{k_2x} + De^{-k_2x}$

3 $\quad Ee^{ik_3x} + Fe^{-ik_3x}$

We can simplify if we only consider waves travelling to the right in region 3 so $F = 0$. We can also consider a specific energy of a particle and its de-Broglie wavelength.

Recall some properties of the wave function
- higher energy – smaller wavelength
- classically forbidden regions – wave functions decay exponentially

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E-V)\psi$$

always towards axis — if $E-V>0$, $\frac{d^2\psi}{dx^2}$ is negative if ψ is >0
positive if ψ is <0

always away from axis — if $E-V<0$ $\frac{d^2\psi}{dx^2}$ is positive (always)
negative

- Wave functions are smooth

$$\psi_1 = \psi_2 \,, \quad \frac{d\psi_1}{dx} = \frac{d\psi_2}{dx} \quad \text{at } x = 0$$

$$\psi_2 = \psi_3 \,, \quad \frac{d\psi_2}{dx} = \frac{d\psi_3}{dx} \quad @ \; x = a$$

$k_1 = \sqrt{2mE}/\hbar$

$k_2 = \sqrt{2m(U-E)}/\hbar$

$k_3 = \sqrt{2mE}/\hbar = k_1$

$$A + B = C + D$$

$$(Aik_1 - Bik_1) = k_2(C - D)$$

$$Ee^{ik_3a} = Ce^{k_2a} + De^{-k_2a}$$

$$ik_3Ee^{ik_3a} = k_2\left(Ce^{k_2a} - De^{k_2a}\right)$$

yikes! These are easier to solve if we let $E > U$ 1st & then relax this condition

we get:

$$Aik_1 - Bik_2 = = ik_2(C - D) = ik_1(A - B)$$

$$A + B = C + D$$

$$Ee^{ik_1a} = Ce^{ik_2a} + De^{-ik_2a}$$

$$Ek_1e^{ika} = k_2\left(Ce^{ik_2a} - De^{-ik_2a}\right)$$

5 equations & 4 unknowns - traditionally we leave A as a variable and look at quantities in relation to A. In this case we are trying to find E^2/A^2

transmitted intensity — incident intensity

EE^*/AA^*

$$k_1(A+B) + k_1(A-B) = k_1(C+D) + k_2(C-D)$$

$$2k_1A = (k_1+k_2)C + (k_1-k_2)D$$

$$k_2 E e^{ik_1a} + k_1 E e^{ik_1a} = k_2 C e^{ik_2a} + k_2 D e^{-ik_2a} + k_2 C e^{ik_2a} - k_2 D e^{-ik}$$

$$(k_2+k_1)Ee^{ik_1a} = 2k_2Ce^{ik_2a}$$

$$(k_2-k_1)Ee^{ik_1a} = 2k_2De^{-ik_2a}$$

or $$C = \frac{(k_2+k_1)}{2k_2} E e^{ik_1a} e^{-ik_2a}$$

$$D = \frac{(k_2-k_1)}{2k_2} E e^{ik_1a} e^{ik_2a}$$

& $$2k_1A = \frac{(k_1+k_2)^2}{2k_2} E e^{ik_1a} e^{-ik_2a} - \frac{(k_2-k_1)^2}{2k_2} E e^{ik_1a} e^{ik_2a}$$

$$A = \frac{(k_1+k_2)^2}{4k_1k_2} E e^{ik_1a} e^{-ik_2a} - \frac{(k_2-k_1)^2}{4k_1k_2} E e^{ik_1a} e^{ik_2a}$$

substitute $k_2 = i\alpha$ (for $E<V$)

$$\frac{AA^*}{EE^*} = \left(\frac{(k_1+i\alpha)^2}{4k_1i\alpha} e^{ik_1a} e^{\alpha a} - \frac{(k_1-i\alpha)^2}{4k_1i\alpha} e^{ik_1a} e^{-\alpha a}\right) \times$$
$$\left(\frac{(k_1-i\alpha)^2}{-4k_1i\alpha} e^{-ik_1a} e^{\alpha a} + \frac{(k_1+i\alpha)^2}{4k_1i\alpha} e^{-ik_1a} e^{-\alpha a}\right)$$

$$= \frac{(k_1^2+\alpha^2)^4}{(4k_1\alpha)^2} e^{2\alpha a} + \frac{(k_1^2+\alpha^2)^4}{(4k_1\alpha)^2} e^{-2a\alpha} - \frac{(k_1+i\alpha)^4}{(4k_1\alpha)^2} - \frac{(k_1-i\alpha)^4}{(4k_1\alpha)^2}$$

note: $(k_1+i\alpha)^4 = k_1^4 - 4i\alpha k_1^3 - 6k_1^2\alpha^2 + 4i\alpha^3k_1 + \alpha^4$

$$(k_1-i\alpha)^4 = k_1^4 + 4i\alpha k_1^3 - 6k_1^2\alpha^2 - 4i\alpha^3k_1 + \alpha^4$$

$$(k_1+i\alpha)^4 + (k_1-i\alpha)^4 = 2(k_1^4 - 6k_1^2\alpha^2 + \alpha^4) = 2(k_1^2+\alpha^2)^2 - 16k_1^2\alpha$$

$$\frac{AA^*}{EE^*} = \left[\left(\frac{(k_1^2+\alpha^2)}{2k_1\alpha}\frac{(e^{\alpha a} - e^{-\alpha a})}{2}\right)^2 + \frac{16k_1^2\alpha^2}{16k_1^2\alpha^2}\right]$$

$$T = \frac{AA^*}{EE^*} = \left[\frac{(2mE + (2m)(V-E))^2}{4(2mE)(2m)(V-E)} \sinh^2(\alpha a) + 1 \right]$$

$$= \left[4\frac{V^2}{4(V-E)E} \sin^2 h(\alpha a) + 1 \right]$$

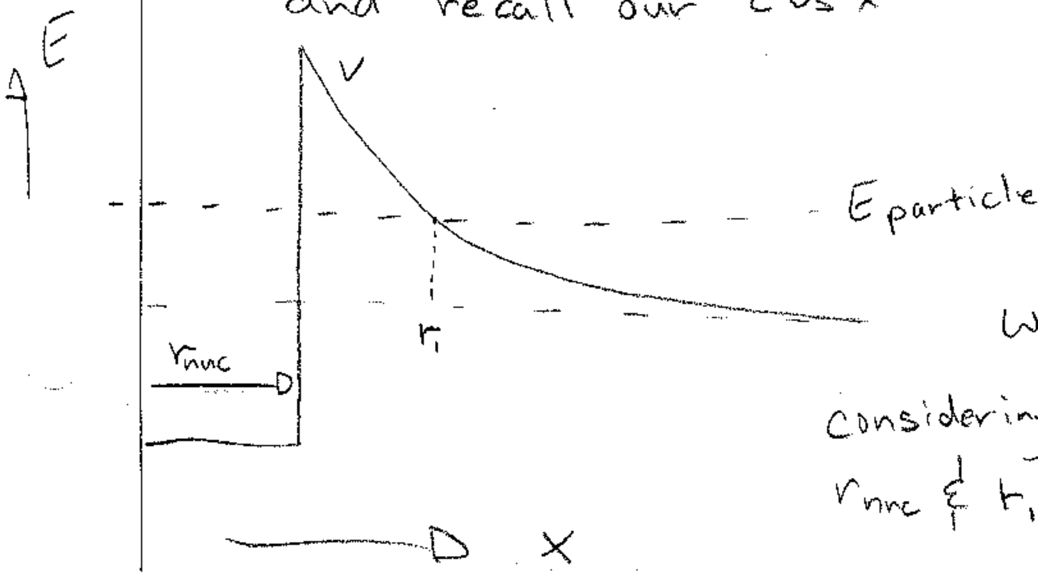
$$\text{Transmission probability} = \left[1 + \frac{\sinh^2(\alpha a)}{4\frac{E}{V}(1-\frac{E}{V})} \right]^{-1}$$

For our alpha decay, we will treat the transmission probability as small with $\sinh^2(\alpha a)$ large

so $$T = 16\frac{E}{V}\left(1-\frac{E}{V}\right)e^{-2\alpha a}$$

To get a feel for the order of magnitude behavior of this, notice, for "reasonable" values of E & V ($E \sim 4$ MeV, $V \sim 30$ MeV) $16\frac{E}{V}(1-\frac{E}{V})$ is essentially $\mathcal{O}(1)$ over the range $\frac{1}{10} = \frac{E}{V}$ to nearly $1 = \frac{E}{V}$, so we'll neglect it. Now consider the remaining term $$T \sim e^{-2\alpha a}$$

and recall our E vs x



We proceed by considering T between r_{nuc} & r_1