Insert after Page 5, Lect 1

Consider errors that are not independent

$$(V-\bar{V}) = \frac{\partial V}{\partial x_1}(x_1-\bar{x}_1) + \frac{\partial V}{\partial x_2}(x_2-\bar{x}_2)$$

$$\langle (V-\bar{V})^2 \rangle = \left|\frac{\partial V}{\partial x_1}\right|^2 \langle (x_1-\bar{x}_1)^2 \rangle + \left|\frac{\partial V}{\partial x_2}\right|^2 \langle x_2-\bar{x}_2 \rangle^2$$

$$\underbrace{+ 2\frac{\partial V}{\partial x_1}\frac{\partial V}{\partial x_2}\langle (x_1-\bar{x})(x_2-\bar{x}) \rangle}$$

this term can be positive or negative. For independent random variables, it should be positive as much as negative so we usually just drop it.

In general, this term is called the covariance

i.e. $$\sigma_V^2 = \left|\frac{\partial V}{\partial x_1}\right|^2 \sigma_{x_1}^2 + \left|\frac{\partial V}{\partial x_2}\right|^2 \sigma_{x_2}^2 + 2\,\mathrm{cov}(x_1, x_2)$$

<u>ex</u> suppose we have 2 signals

$s_1 = s_0 + \delta(t) + \sigma_1$ σ_1 is a random source of noise

$s_2 = s_0 + \delta(t) + \sigma_2$ σ_2 is a random source of noise

$\delta(t)$ is some noise source common to both signals but uncorrelated with σ_1 or σ_2

of course you would subtract these 2 signals

$s_{tot} = 2s_0 + ?$

$$\langle (s_1-\bar{s}_1)(s_2-\bar{s}_2) \rangle = \langle (\delta(t)+\sigma_1)(\delta(t)+\sigma_2) \rangle$$

if $s_{tot} = s_1 - s_2$

$$= \underbrace{\langle \delta(t)\sigma_2 \rangle + \langle \delta(t)\sigma_1 \rangle}_{\text{these terms are 0 if } \sigma \text{ \& } \delta \text{ are uncorrelated}} + \langle \delta(t)\delta(t) \rangle + \cancel{\langle \sigma_1\sigma_2 \rangle}^{0}$$

$$\sigma_{tot}^2 = \langle \sigma_1^2 \rangle + 2\langle \sigma_1 \delta(t) \rangle + \langle \delta(t) \rangle^2 + \langle \sigma_2^2 \rangle + 2\langle \sigma_2 \delta(t) \rangle + \langle \sigma_2^2 \rangle$$
$$- 2\left(\langle \delta(t)\sigma_1 \rangle + \langle \delta(t)\sigma_2 \rangle + \langle \delta(t)^2 \rangle\right) = \sigma_1^2 + \sigma_2^2$$

if $s_{tot} = s_1 + s_2$

$$\sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2 + 4\langle \delta(t)^2 \rangle + 4\langle \delta(t)\sigma_1 \rangle + 4\langle \delta(t)\sigma_2 \rangle$$

if $\delta(t)$ dominates $\sigma_{tot} \sim 2\delta(t)$

ex Fit to a straight line $y = a + bx$

construct $\chi^2 = \sum_i \frac{(y_i - (a+bx_i))^2}{\sigma_i^2}$ (σ_i is in general the error on y)

want to minimize χ^2 w.r.t. the parameters

$$\frac{\partial \chi^2}{\partial a} = \sum_i -2\frac{(y_i-(a+bx_i))}{\sigma_i^2} = 0$$

$$\frac{\partial \chi^2}{\partial b} = \sum_i -2x_i\frac{(y_i-(a+bx_i))}{\sigma_i^2}$$

gives simultaneous equations for a & b

organize terms $S = \sum_i \frac{1}{\sigma_i^2}$, $S_x = \sum_i \frac{x_i}{\sigma_i^2}$

$S_y = \sum_i \frac{y_i}{\sigma_i^2}$, $S_{xx} = \sum_i \frac{x_i^2}{\sigma_i^2}$

$S_{xy} = \sum_i \frac{x_i y_i}{\sigma_i^2}$

$$\Rightarrow \quad aS + bS_x - S_y = 0$$
$$aS_x + bS_{xx} - S_{xy} = 0$$

inverting & multiplying by the inverse gives us the params.

(note how we constructed this to be symmetric)

$$\begin{pmatrix} S & S_x \\ S_x & S_{xx} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} S_y \\ S_{xy} \end{pmatrix}$$

$$\left\{ \begin{pmatrix} m & n \\ o & p \end{pmatrix}^{-1} = \frac{1}{mp - on}\begin{pmatrix} p & -n \\ -o & m \end{pmatrix} = \frac{1}{DET}\begin{pmatrix} p & -n \\ -o & m \end{pmatrix} \right\}$$

for us $DET = S S_{xx} - S_x^2$

$$\frac{1}{DET}\begin{pmatrix} S_{xx} & -S_x \\ -S_x & S \end{pmatrix}\begin{pmatrix} S & S_x \\ S_x & S_{xx} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{DET}\begin{pmatrix} S_{xx} & -S_x \\ -S_x & S \end{pmatrix}\begin{pmatrix} S_y \\ S_{xy} \end{pmatrix}$$

$$a = (S_{xx}S_y - S_x S_{xy})/DET$$
$$b = (-S_x S_y + S S_{xy})/DET$$

Proceed as before to get errors

$$\sigma_a^2 = \langle (a-\bar{a})^2 \rangle = \sum_i \left|\frac{\partial a}{\partial y_i}\right|^2 \sigma_i^2$$

$$\sigma_b^2 = \langle (b-\bar{b})^2 \rangle = \sum_i \left|\frac{\partial b}{\partial y_i}\right|^2 \sigma_i^2$$

$$\mathrm{cov}(a,b) = \sigma_{ab} = \langle (b-\bar{b})(a-\bar{a}) \rangle = \sum_i \frac{\partial a}{\partial y_i}\frac{\partial b}{\partial y_i}\sigma_i^2$$

$$\frac{\partial a}{\partial y_i} = \frac{S_x x_i - S_{xx}}{\sigma_{y_i}^2 \, DET}$$

picks out particular i for pieces with a y

$$\frac{\partial b}{\partial y_i} = \frac{S_x - S x_i}{\sigma_{y_i}^2 \, DET}$$

$$\sigma_a^2 = \sum_i \sigma_i^2 \left(\frac{S_x x_i - S_{xx}}{\sigma_i^2 \, DET}\right)^2 = \sum_i \frac{S_x^2 x_i^2}{\sigma_i^2 DET^2} - 2\sum_i \frac{S_x x_i S_{xx}}{\sigma_i^2 DET} + \sum_i \frac{S_{xx} S_{xx}}{\sigma_i^2 DET^2}$$

$$= \frac{S_x^2 S_{xx}}{DET^2} - \frac{2 S_x^2 S_{xx}}{DET^2} + \frac{S S_{xx}^2}{DET^2} = \frac{S S_{xx}^2 - S_x^2 S_{xx}}{DET^2}$$

$$= \frac{S_{xx}(S S_{xx} - S_x^2)}{DET^2} = S_{xx}/DET$$

$DET = S S_{xx} - S_x^2$

$$\sigma_b^2 = \sum_i \sigma_i^2 \left(\frac{S_x - S x_i}{\sigma_{y_i}^2 \, DET}\right)^2 = \sum_i \frac{S_x^2}{\sigma_i^2 DET^2} - 2\sum_i \frac{S_x S x_i}{\sigma_i^2 DET^2} + \sum_i \frac{S^2 x_i^2}{\sigma_i^2 DET^2}$$

$$= \frac{S S_x^2}{DET^2} - 2\frac{S S_x^2}{DET^2} + \frac{S^2 S_{xx}}{DET^2} = \frac{S S_{xx} - S S_x^2}{DET^2} = S/DET$$

$$\sigma_{ab}^2 = \sum_i \sigma_i^2 \left(\frac{S_x x_i - S_{xx}}{\sigma_i^2 \, DET}\right)\left(\frac{S_x - S x_i}{\sigma_i^2 \, DET}\right)$$

$$= \sum_i \frac{S_x^2 x_i}{\sigma_i^2 DET^2} - \sum_i \frac{S S_x x_i^2}{\sigma_i^2 DET^2} - \sum_i \frac{S_{xx} S_x}{\sigma_i^2 DET^2} + \sum_i \frac{S S_{xx} x_i}{\sigma_i^2 DET^2}$$

$$= \frac{S_x^3}{DET^2} - \frac{S_x S S_{xx}}{DET^2} - \frac{S S_{xx} S_x}{DET^2} + \frac{S S_{xx} S_x}{DET^2} = -\frac{S_x (S S_{xx} - S_x^2)}{DET^2}$$

$$= -S_x/DET$$

Notice

From our matrix equation

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} \begin{pmatrix} S_y \\ S_{xy} \end{pmatrix}$$

& you can imagine generalizing to many dimensions

$$\begin{pmatrix} S_{00} & S_{10} & S_{20} & \cdots & S_{n0} \\ S_{01} & S_{11} & S_{21} & & \\ S_{02} & S_{12} & S_{22} & & \\ \vdots & & & \ddots & \\ S_{0n} & & & & S_{nn} \end{pmatrix}$$

where 0 refers to
x_i^0 term
1 to x_i^1
2 to x_i^2
$S_{11} \Rightarrow x_i^1 x_i^1$

and the error matrix is just the inverse
(if the inverse exists)

From $y = a + bx$

$$\sigma_y^2 = \left|\frac{\partial y}{\partial a}\right|^2 \sigma_a^2 + \left|\frac{\partial y}{\partial b}\right|^2 \sigma_b^2 + 2\left(\frac{\partial y}{\partial a}\right)\left(\frac{\partial y}{\partial b}\right) \mathrm{cov}(a,b)$$

$$= \sigma_a^2 + x^2\sigma_b^2 + 2(1)(x)\,\mathrm{cov}(a,b)$$

$$= \sigma_a^2 + x^2\sigma_b^2 + 2x\,\mathrm{cov}(a,b)$$