Damped Oscillation Solution

**OverDamped Oscillation Solution**

The last case has $\beta^2 - \omega_0^2 > 0$. In this case we define another real frequency $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$. In terms of this frequency, the overdamped solution is

$$x(t) = [A_1 \exp(\omega_2 t) + A_2 \exp(-\omega_2 t)] \exp(-\beta t)$$

As in the other two solutions you see the envelope $\exp(-\beta t)$ function driving the displacement towards 0 at larger times.

As you will see, the overdamped solution starts out with some initial displacement either positive or negative, and then goes to zero displacement with increasing time without ever crossing the horizontal axis into an opposite sign displacement.

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**Energy Loss in Underdamped Oscillation, see example provided**

The basic fact of damped oscillation is that there is a friction term which is dissipating energy. Mechanical energy (Kinetic + Potential) is *not* conserved in any damped motion. If the damping factor is not too large, meaning $\beta/\omega_1 << 1$ or equivalently $\omega_0 >> \beta$, then one can write the energy function of time as

$$E(t) = \frac{1}{2} kA^2 \exp(-2\beta t)$$

Note that the energy is dropping faster than the envelope function for the motion, which drops as $\exp(-\beta t)$. For this light damping case we can compute the time it takes for the energy to drop to a factor $1/e$ of its original value. We call that time $\tau$ the *characteristic time* for which we have

$$E(t = \tau) = \frac{E_0}{e} = E_0 \exp{-2\beta \tau} \implies \tau = \frac{1}{2\beta}$$

You can also compute that the fractional rate of energy loss $(dE/dt)/E$ as

$$\frac{1}{E} \frac{dE}{dt} = -2\beta$$
Energy Damping

Quality Factor

We have seen that for damped motion the energy must decrease with time. We can write

\[ E(t) = E_0 \exp(-2\beta t) \]

where \( \beta \) is the damping factor. If there is no damping, then there is no energy loss. For oscillating systems with damping there is a dimensionless descriptive parameter called the \textit{Quality Factor} \( Q \) defined as follows

\[ Q \equiv \frac{\text{Energy stored in oscillator}}{\text{Average energy dissipated per radian}} \]

Since the energy stored in a damped oscillator is decreasing, the above is a somewhat ill-defined definition. You can think about \( Q \) as the energy stored at a given time divided by the energy lost during the next radian worth of time. Since \( \omega_1 \) gives you the radians per second, you can figure out how much time it takes to cover one radian of oscillation.

We can obtain \( dE/dt \) by differentiating the expression for \( E(t) \)

\[ \frac{dE}{dt} = -2\beta E_0 \exp(-2\beta t) \]

Now we can compute the energy lost during one radian of oscillation as

\[ \Delta E = -\frac{dE}{dt} \Delta t = 2\beta E_0 \exp\left(-\frac{2\beta t}{\omega_1}\right) \]

\[ Q = \frac{E_{\Delta E}}{\Delta E} = \frac{\omega_1 E_0 \exp(-2\beta t)}{2\beta E_0 \exp(-2\beta t)} = \frac{\omega_1}{2\beta} \]

You should remember that \( \omega_1 = \sqrt{\omega_0^2 - \beta^2} \). And \textit{light damping} means that \( \beta << \omega_0 \). So for a lightly damped system

\[ Q \approx \frac{\omega_0}{2\beta} \]

Some interesting physical systems such as loudspeakers have low \( Q \) values, in the range 5 to 100. You don’t want to keep hearing the same sound vibrations long after the original musical note was struck. On the other hand, something like a tuning fork where you want to keep hearing the original vibration, the \( Q \) factor can be 1,000. And in the electrical analogs of mechanical oscillators, one can have \( Q \) values ranging from 10,000 to \( 10^{14} \) as in superconducting resonators.
Forced Oscillation

Newton’s Law for Forced Oscillation

The last major oscillation topic is force oscillation. We consider adding an external time-dependent so-called driving force $F_d(t)$ to the restoring force $-kx$ and to the linear resistive force $-bx$. So our Newton’s Second Law differential equation becomes

$$m\ddot{x} + b\dot{x} + kx = F_d(t)$$

Although $F_d(t)$ in principle could have any kind of time-dependence, it is most instructive to consider $F_d(t) = F_0 \cos(\omega t + \phi_0)$ as the time dependence as the first example. Besides being easy to solve, a cosine or sine dependence can be attributed to any periodic driving force (Fourier series expansion).

Solution to driven oscillator

Assuming a cosine dependent external force, we have to solve the following

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t + \phi_0)$$

where $\omega$ is yet another angular frequency in addition to our previous $\omega_0$ and $\omega_1$. The solution to this type of differential equation, with a non-zero right hand side, is well-known in mathematics. The solution consists of two separate functions summed together. The first function $x_h(t)$ is called the homogeneous solution which solves the differential equation having a zero on the right hand side. This solution also includes the two integration constants. The second function $x_i(t)$ is called the particular or inhomogeneous function, which when substituted in the differential equation, will produce the right hand side non-zero term. The particular solution does not have the constants of integration determined from the initial conditions.

Review of Homogeneous Solution

The homogeneous or general solution has the following explicitly form

$$x_h(t) = A_h \exp(-\gamma t) \cos(\omega_1 t + \phi_h)$$

The amplitude constant $A_h$ and the phase angle constant $\phi_h$ are determined from two initial conditions, such as the initial position $x(t = 0)$ and the initial speed $\dot{x}(t = 0)$. The envelope factor $\exp(-\gamma t)$ causes the homogeneous solution to decay to zero over time.
Forced Oscillation

Particular Solution
For the forced oscillator differential equation
\[ m\ddot{x} + b\dot{x} + kx = F_d(t) \]
we have the solution
\[ x(t) = x_h(t) + x_i(t) \]

Just before we showed the form of the homogeneous solution \( x_h(t) \). Now we derive the form of the inhomogeneous or particular solution \( x_i(t) \). The particular solution is dependent on the nature of the driving force \( F_d(t) \). We assume a driving force of the form \( F_d(t) = F_0 \cos(\omega t) \). We have dropped the phase angle \( \phi_0 \) on the premise that we choose to call \( t = 0 \) when \( F = F_0 \). Then, for \( x_i(t) \) we adopt a trial solution, for which the sensible thing to do is to assume a cosine function

\[ \text{trial solution} \quad x_i(t) = A \cos(\omega t - \phi) \]

where \( A \) and \( \phi \) are to be determined, if the trial solution shows promise of being right.

Working out the trial solution
When we substitute in the trial solution for the inhomogeneous term, we will obtain this result
\[ -m\omega^2 A \cos(\omega t - \phi) - b\omega \sin(\omega t - \phi) + kA \cos(\omega t - \phi) = F_0 \cos \omega t \]
We need to find values of \( A \) and \( \phi \), which are constant, and which satisfy this equation. The several steps to do this search. Here we simply quote the results.

\[ A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \]
\[ \phi(\omega) = \tan^{-1} \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \]

Notice that the \( \omega \) dependence of these constants \( A \) and \( \phi \) is emphasized by writing the constants as \( A(\omega) \) and \( \phi(\omega) \). The particular solution depends on the driving force.
Forced Oscillation

Complete Solution for Forced Oscillation
The complete solution for the forced oscillation problem looks like

\[ x(t) = x_h(t) + x_i(t) = A_h \exp(-\gamma t) \cos(\omega_1 t + \phi_h) + \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2}} \cos(\omega t - \phi) \]

with

\[ \phi(\omega) = \tan^{-1}\frac{2\gamma \omega}{\omega_0^2 - \omega^2} \]

Understanding the complete solution
Notice that there are three frequency variables above

1) \( \omega_0 = \sqrt{k/m} \) is the free oscillation frequency
2) \( \omega_1 = \sqrt{\omega_0^2 - \gamma^2} \) is the damped oscillation frequency, and
2) \( \omega \) is the external driving force frequency

As you might expect, there are different and interesting physics situations depending on how the value of \( \omega \) compares to the value of \( \omega_0 \).

When the frequency \( \omega \) of the driving force is much greater than the free oscillation frequency, then the resulting motion is largely dominated by the driving force. On the other hand, when the driving force frequency is much less than the free oscillation frequency, the resulting motion looks much like the free oscillation case with some slight change. The \( \omega >> \omega_0 \) case is called *distortion* and the \( \omega << \omega_0 \) case is called *modulation*. 
Forced Oscillation

Resonance Case
The other interesting case is the one for which $\omega \approx \omega_0$. If you look at the amplitude term for the inhomogeneous solution, it has the form

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

If you do the following derivative completely

$$\frac{dA}{d\omega} = 0$$

you can determine the driving frequency $\omega_r$ for which the amplitude $A(\omega_r)$ is a maximum. The result for the resonant frequency $\omega_r$ is

$$\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$$

Depending on the value of $\gamma$, the amplitude $A$ can become very large. In fact, if $\gamma = 0$ and $\omega = \omega_0$, then the amplitude in force oscillation becomes infinite.

In real physical systems, there is always some damping. And even for low damping cases, if the oscillation amplitude gets too large, then the simple linear restoring force approximation is no longer valid. However, it can still happen for a real system that an external driving force cause a very large amplitude of vibration. Hence the caution that when a group of soldiers marches over a bridge that they should break ranks and stop marching in formation.