Problem of Falling Mass, Rotating Pulley, and Rolling Mass

Qualitative Description

Problem 10.83 is not easy to solve, but it provides a good analysis of how linear accelerated motion is coupled into rotational acceleration motion. Please print out a copy of the illustration provided at the class web site *Illustration for solving problem 10.83*.

First, you should get a qualitative understanding of the entire motion. The vertically suspended mass M is accelerating downward because of gravity. The accelerating force for Newton's Second Law is the downward weight force of the mass $M\vec{g}$ less the upward tension force $\vec{T}_3$ from the connecting string. That string also exerts a downward tension force $\vec{T}'_3$ on the right side of the pulley. The prime superscript means that this force is equal in magnitude but opposite in direction of the $\vec{T}_3$ force. This $\vec{T}'_3$ force acts to turn the pulley clockwise.

On the top of pulley a horizontal force $\vec{T}'_2$, which is directed to the left, acts to rotate the pulley counterclockwise. The $\vec{T}'_3$ force acting down will turn out to be bigger than the $\vec{T}'_2$ force acting to the left. Both forces are at the same distance R from the center of the pulley. Therefore, the pulley will experience a net clockwise torque, causing it to have a rotational acceleration in the clockwise direction. This is as you would expect for a real falling mass.

For the big disk of radius $2R$ and mass M, there are also two forces acting. One force acting to the right at the center of mass of the disk is labeled $\vec{T}_2$. Again, this force is equal and opposite to the $\vec{T}'_2$ force. The $\vec{T}_2$ force acts to accelerate the center-of mass of the big disk to the right, with the same linear acceleration (string cannot stretch) as the vertically falling mass.

For the big mass there is also a force of friction $\vec{T}_1$ acting to the left which causes the big disk to have a rotational acceleration about the center-of-mass in the clockwise direction, again as you would expect. *If there were not friction between the table and the big disk, then the big disk could not rotate! Without friction on the table the big disk can only slide along the table.*

Finally, the big mass and the falling mass have the same acceleration a since they are connected by a string. The rotational acceleration of the pulley α_2 and that of the rolling mass α_1 are connected to the linear acceleration by the no-slip constraining equations $a = R\alpha_2$ and $a = 2R\alpha_1$, respectively. From these constraint equations, you can deduce that the smaller radius pulley has twice the rotational acceleration compared to the larger radius rolling mass.

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Algebraic Solution

From the preceding discussion, we can say that the falling mass M has a linear acceleration a, the pulley has a rotational acceleration α_2, and the rolling mass has *both* a linear acceleration a of its center-of-mass, and a rotational acceleration α_1 around its center-of-mass.

We can write two Newton's Second Law equations for the linear acceleration and two Newton's Second Law equations for the rotational acceleration in terms of the forces mentioned above, and the acceleration variables. Just as in the illustration, these four equations are

$$T_1 \times 2R = I_1\alpha_1 = I_1(\frac{a}{2R}) \tag{1}$$

$$T_2 - T_1 = Ma \tag{2}$$

$$(T_3 - T_2)R = I_2\alpha_2 = I_2(\frac{a}{R}) \tag{3}$$

$$Mg - T_3 = Ma \tag{4}$$

1) The first equation relates the torque of the T_1 force about the center of the rolling mass to the rotational acceleration of that mass.

2) The second equation gives the net force on the rolling mass as equal to its mass times its linear acceleration.

3) The third equation gives the net torque on the pulley about its center as equal to the product of its moment of inertia and its rotational acceleration.

4) The fourth equation gives the net force on the vertically suspended mass as equal to the product of its mass and its linear acceleration.

These four equations can be solved in turn to produce

$$T_1 = \frac{Ma}{2} \quad , \quad T_2 = \frac{3Ma}{2} \quad , \quad T_3 = 2Ma$$

and finally

$$a = \frac{g}{3}$$

With this expression for a we can see that $T_3 = Mg/2$, $T_2 = 3Mg/2$, and $T_1 = Mg/6$. The lengths of the force vectors in the illustration are scaled appropriately. So you should be able to infer from that illustration the correct directions in which each of the three objects will be moving.