Projectile at an Angle ϕ Above Inclined Plane of Angle θ

▪ Part A

The projectile is being fired at an angle $\phi+\theta$ above the horizontal. So the trajectory equation is

$y(x) = \tan(\phi+\theta)x - gx^2/(2v_0^2\cos^2(\phi+\theta))$ where v_0 is the speed.

Now if the projectile hits the plane at coordinates (x,y), then we know the expression for $\tan\theta$

$\tan\theta = y/x$ which gives $y = x \tan\theta$. We can set these two expression for y equal to each other

$x \tan\theta = \tan(\phi+\theta)x - gx^2/(2v_0^2\cos^2(\phi+\theta))$

This equation has a solution at x = 0, corresponding to y = 0, meaning when the projectile was fired.

There is a second solution when x is not 0, which we can derive by first dividing both sides by x

$\tan\theta = \tan(\phi+\theta) - gx/(2v_0^2\cos^2(\phi+\theta))$ which leads to this equation for x on the left side

$x = [\ \tan(\phi+\theta) - \tan\theta]2v_0^2\cos^2(\phi+\theta)$

Last, we can divide by $\cos\theta$ to get the distance along the inclined plane (range)

$x/\cos\theta = [\ \tan(\phi+\theta) - \tan\theta]2v_0^2\cos^2(\phi+\theta)/\cos\theta$

■ Part B

To get the maximum range, we need to take the derivative with respect to the angle ϕ of the right hand side of the last equation, and set that derivative to 0. There is a commercial computer program called *Mathematica* which can do derivatives and algebraic operations easily, although there is a learning curve of a few days time on getting the *Mathematica* syntax correct. We can simplify the problem by dropping the factor $2v_0^2/\cos\theta$ which do not depend upon the angle ϕ. So we are left with the task of taking the derivative of the expression $[\tan(\phi+\theta) - \tan\theta]\cos^2(\phi+\theta)$. In the following, the text in **RED** is an input command to Mathematica, while the text in **MAGENTA** is the output result of that command. First we find the derivative with respect to the angle ϕ, and the *Mathematica* command is

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In[22]:= D[(Tan[ϕ + θ] - Tan[θ]) * (Cos[ϕ + θ])^2, ϕ]

Out[22]= 1 - 2 Cos[θ + ϕ] Sin[θ + ϕ] (-Tan[θ] + Tan[θ + ϕ])
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The above is the derivative result, and we can simplify it using trig identities to obtain

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In[23]:= FullSimplify[
           1 - 2 Cos[θ + ϕ] Sin[θ + ϕ] (-Tan[θ] + Tan[θ + ϕ])]

Out[23]= Cos[θ + 2 ϕ] Sec[θ]
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Lastly, we can set the above result equal to 0 in order to find the value of ϕ which gives a maximum. By simple inspection you can see that $\theta+2\phi=\pi/2$ is the correct answer since it makes the cosine term to be 0.

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In[24]:= Solve[Cos[θ + 2 ϕ] Sec[θ] == 0, ϕ]
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Out[24]= $\left\{\left\{\phi \to \frac{1}{2}\left(-\frac{\pi}{2} - \theta\right)\right\}, \left\{\phi \to \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right\}\right\}$

The second (positive) solution $\phi = \pi/4 - \theta/2$ is what we are looking for, since negative angles would not have any physical meaning in this problem.