REVIEW: (Chapter 8) LINEAR MOMENTUM and COLLISIONS

The first new physical quantity introduced in Chapter 8 is **Linear Momentum**
Linear Momentum can be defined first for a particle and then for a system of particles or an extended body. It is just the product of mass and velocity, and is a vector in the same direction as the velocity:

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \quad \text{particle}$$

$$\vec{\mathbf{P}} = M\vec{\mathbf{v}}_{cm} \quad \text{system of particles} \qquad \vec{\mathbf{v}}_{cm} \equiv \text{center-of-mass velocity}$$

Why have this momentum quantity? In fact it was Newton himself who introduced the quantity in his version of Newton's Second Law. For the case of a particle one has:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$$

$$\Longrightarrow \vec{\mathbf{F}} = \frac{d(m\vec{\mathbf{v}})}{dt} = m\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}}$$

Here we are making use of the fact that the mass m of a particle does not change with time. The same derivation can be made for a system of particles, or an extended body, as long as we always include all the mass.

$$\vec{\mathbf{F}}_{ext} = \frac{d\vec{\mathbf{P}}}{dt}$$

$$\Longrightarrow \vec{\mathbf{F}}_{ext} = \frac{d(M\vec{\mathbf{v}}_{cm})}{dt} = M\frac{d\vec{\mathbf{v}}_{cm}}{dt} = M\vec{\mathbf{a}}_{cm}$$

Conservation of Linear Momentum

The important use of Linear Momentum comes about when we consider the special case when there is no net force acting. This defines an **isolated system**. In that case, the left hand sides of the two above equations are zero. Therefore, the linear momentum of the particle, or of the system of particles, is constant.

$$F = 0 \Longrightarrow \vec{\mathbf{p}} = \ \textbf{constant} \ \text{ or } \ \vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f$$

$$F_{ext} = 0 \Longrightarrow \vec{\mathbf{P}} = \ \textbf{CONSTANT} \ \text{ or } \ \vec{\mathbf{P}}_i = \vec{\mathbf{P}}_f$$

THE CONSERVATION OF ENERGY LAW AND THE CONSERVATION OF MOMENTUM LAW ARE THE TWO MOST IMPORTANT LAWS OF PHYSICS. THESE TWO LAWS ARE THE FOUNDATION OF SCIENCE.

EXAMPLE of LINEAR MOMENTUM CONSERVATION

One example of linear momentum conservation involves the recoil of a cannon (or a rifle) when a shell is fired.

A cannon of mass $M = 3000$ kg fires a shell of mass $m = 30$ kg in the horizontal direction. The cannon recoils with a velocity of 1.8 m/s in the $+\hat{\mathbf{i}}$ direction. What is the velocity of the velocity of the shell just after it leaves the cannon ball?

Remember that we have to deal with **isolated** or self-contained systems. In this example the **isolated system** is the cannon plus the shell, not just the cannon by itself of the shell by itself. The explosion which fires the shell is an INTERNAL force, so it does not enter into the problem. There are no EXTERNAL forces acting in the horizontal direction, so linear momentum is conserved in the horizontal direction

$$\vec{\mathbf{P}}_i = \vec{\mathbf{P}}_f$$

The initial linear momentum $\vec{\mathbf{P}}_i = 0$ because nothing is moving.
The final linear momentum $\vec{\mathbf{P}}_f = 0$ also, but it can be expressed as the sum of the linear momenta of the cannon and the shell:

$$\vec{\mathbf{P}}_i = 0 = \vec{\mathbf{P}}_f = M\vec{\mathbf{V}} + m\vec{\mathbf{v}}$$

Here $\vec{\mathbf{V}}$ is the velocity of the cannon and $\vec{\mathbf{v}}$ is the velocity of the shell.

Clearly $\vec{\mathbf{V}}$ and $\vec{\mathbf{v}}$ are in opposite directions, and

$$\vec{\mathbf{v}} = -\frac{M\vec{\mathbf{V}}}{m} \Longrightarrow \vec{\mathbf{v}} = -\frac{3000 \cdot 1.8}{30}\,\hat{\mathbf{i}} = -180\,\hat{\mathbf{i}} \text{ m/s}$$

Decay of Subatomic Particles

Another example of conservation of momentum is the decay of an isolated subatomic particle such as a *neutral kaon* written symbolically as K^0. A neutral kaon decays into two other subatomic particles called charged pions, symbolized as π^+ and π^-. The *decay equation* is written as

$$K^0 \rightarrow \pi^+ + \pi^-$$

By conservation of momentum we can easily prove that the two pions have equal and opposite momenta.

REVIEW: Impulse of a Force

We define another vector physical quantity called the **Impulse of a Force**.

In the simplest case, if a constant force **F** acts over a short period of time Δt, then the impulse of that force is equal to the product of the force and the length of time over which it acts. The impulse vector is denoted by the symbol $\vec{\mathbf{J}}$

$$\vec{\mathbf{J}} = \vec{\mathbf{F}}\Delta t \quad \text{(constant force } \mathbf{F}\text{)}$$

$$\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}} \quad \text{(is proved below from Newton's Second Law}$$

If the force is not constant, then the definition of impulse requires an integral

$$\vec{\mathbf{J}} \equiv \int \vec{\mathbf{F}}(t)dt = \Delta\vec{\mathbf{p}}$$

The impulse calculation is useful in determining how much force or momentum is involved in violent collisions lasting very short periods of time.

Impulse Worked Example, compare page 252

A soccer ball of mass 0.45 kg is traveling horizontally to the left at a speed of 20 m/s. A player kicks the soccer ball such that it acquires a speed of 30 m/s at an angle 45^{O} above the horizontal traveling to the right. If the player's foot was in contact with ball for $\Delta t = 0.010$ seconds, what was the magnitude and direction of the force exerted by her foot?

Solution We use the impulse formula $\vec{J} = \overline{\vec{F}}\Delta t = \Delta\vec{p} = m\Delta\vec{v}$ in two dimensions:

$$\text{Momentum change in two dimensions, units of kg-m/s} \quad \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

$$= m(\vec{v}_f - \vec{v}_i) = 0.40(30\cos 45\,\hat{\mathbf{\i}} + 30\sin 45\,\hat{\mathbf{j}} - (-20)\,\hat{\mathbf{\i}}) = 16.5\,\hat{\mathbf{\i}} + 8.5\,\hat{\mathbf{j}} \text{ kg-m/s}$$

$$\overline{\vec{F}}\Delta t = 16.5\,\hat{\mathbf{\i}} + 8.5\,\hat{\mathbf{j}} \text{ kg-m/s} \Longrightarrow \overline{\vec{F}} = \frac{16.5\,\hat{\mathbf{\i}} + 8.5\,\hat{\mathbf{j}}}{0.010} \text{ kg-m/s}^2 \Longrightarrow 1650\,\hat{\mathbf{\i}} + 850\,\hat{\mathbf{j}} \text{ N}$$

We can work out that the magnitude and angle of this average force

$$\overline{F} = \sqrt{(1650)^2 + (850)^2} = 1,900 \text{ N, and } \; \theta_F = \tan^{-1}(850/1650) = 27^{\text{O}}$$

If the kicker has a mass of 50 kg (about 110 pounds), then she is exerting a force of almost 4 times her own weight on the ball. Note that the force of her kick is at a smaller angle than the eventual direction of the ball after the kick.

Supplemental question: What is the magnitude and direction of the average force of the soccer field being exerted on her *other* foot while she is kicking the soccer ball?

REVIEW: Collisions Between Two Isolated Particles

Constant Momentum for an Isolated System

The previous example involved essentially just one particle, the car. The wall was fixed there as a device for exerting a constant force during the collision. A more complex example can be studied when two particles collide. We first make the approximation that the two particles are subjected to no *external* forces. The only forces being exerted are the forces *between* the two particles. In fact these are the *action* and the *reaction* forces which we have seen in discussing Newton's third law.

We can prove that if two particles form an isolated system unaffected by any other particles, then the **vector sum of the momentum of each particle remains constant**. We call the vector sum of the two particle momenta the system momentum P

$$\vec{\mathbf{P}} \equiv \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2$$

$$\vec{\mathbf{P}} = \text{ CONSTANT if there are no external forces}$$

Collision between Two Particles in an Isolated System

When two particles in an isolated system collide, the total momentum of the system is the same after the collision as it was before the collision. The total momentum remains equal to the initial momentum:

$$m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f}$$

$$\vec{\mathbf{p}}_{1i} + \vec{\mathbf{p}}_{2i} = \vec{\mathbf{p}}_{1f} + \vec{\mathbf{p}}_{2f}$$

This principle of the conservation of momentum is one of the strongest in all of physics. Even if there are internal, frictional forces acting which decrease the total mechanical energy, it is nonetheless still true that *the total momentum of an isolated system at all times equals the initial momentum.*

The principle of momentum conservation also applies to an isolated system which suddenly decomposes into two or more interacting pieces. For example a rifle firing a bullet, a cannon firing a cannonball etc. We first define two basic types of collision and then consider each of these basic types in more detail.

Two Basic Types of Two-Particle Collisions Conserving Momentum

The typical situation in momentum conservation involves two particles in the initial system with one or both of these having a velocity. These two particles collide where again only internal forces act, and the particles separate with certain final velocities. Conservation of momentum enables us to relate the final velocities to the initial velocities. There are two basic types of collision:
1) **Elastic Collisions**, and 2) **Inelastic Collisions**

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1) **Elastic Collision**

An **elastic collision** is one in which the total kinetic energy of the two particles is the same after the collision as it was before the collision. Examples of elastic collisions are those between billiard balls, between masses and springs, and those involving rubber or tennis balls.

For **elastic collisions** one can write not only the momentum conservation equation, but also one can write a kinetic energy conservation equation:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \qquad \text{(momentum conserved)}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \text{(kinetic energy conserved)}$$

By combining these two equations one can achieve the general result that

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

The relative velocity of approach is the negative of the relative velocity of separation.

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2) **Inelastic and Perfectly Inelastic Collisions**

If there are very strong frictional and deformation forces, then kinetic energy will no longer be conserved and instead one will have an **inelastic collision.** The limiting case of an **inelastic collision** is one in which the two particles fuse during the collision, and travel together afterwards with the same final velocity $v_{1f} = v_{2f} \equiv v_f \quad \Longrightarrow$ the **perfectly inelastic collision**

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_f + m_2 v_f \quad \text{(particles with same final velocity)}$$

$$\Longrightarrow v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} \quad \text{(perfectly inelastic collision)}$$

Example of Perfectly Inelastic Collision

A Cadillac with a mass of 1800 kg, while stopped at a traffic light, is rear ended by a Volkswagen with a mass of 900 kg traveling at 20 m/s. After the collision both cars are completely entangled, and slide into the intersection. What is their velocity after the collision ?

The relevant equation for a completely inelastic collision is

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} \quad \text{(perfectly inelastic collision)}$$

Take the Cadillac to be m_1, and the Volkswagen to be m_2. In this case we have $v_{1i} = 0$ so

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} = v_f = \frac{m_2 v_{2i}}{m_1 + m_2} = \frac{900 \cdot 20}{1800 + 900} = 6.67 \text{ m/s}$$

We can compute the change in kinetic energy as follows

$$K_i = \frac{1}{2}(m_1 v_{1i}^2 + m_2 v_{2i}^2) = \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2}(900)(20)^2 = 180,000 \text{ Joules}$$

$$K_f = \frac{1}{2}(m_1 v_{1f}^2 + m_2 v_{2f}^2) = \frac{1}{2}(m_1 + m_2) v_f^2 = \frac{1}{2}(900 + 1800)(6.67)^2 = 60,000 \text{ Joules}$$

$$\Delta K \equiv K_f - K_i = 60,000 - 180,000 = -120,000 \text{ Joules}$$

Where did all this kinetic energy go ??

Other types of perfectly inelastic collisions include bullets fired into blocks of wood after which the bullet is slowed down and stopped inside the wood.

Elastic Collisions in One Dimension

The opposite extreme from a perfectly inelastic collision is a perfectly elastic collision where the kinetic energy is conserved. So one has two equations with which to solve problems. By combining those two equations (conservation of momentum and conservation of kinetic energy) one can arrive at a third equation which gives

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

The relative velocity of approach is the negative of the relative velocity of separation

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Example of Elastic Collision

Two billiard balls have velocities of +2.0 and −0.5 m/s before they meet in a head–on collision. What are their final velocities ?

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

$$2 - (-0.5) = -(v_{1f} - v_{2f}) \Longrightarrow v_{2f} = 2.5 + v_{1f}$$

Now substitute this into the general conservation of momentum equation, realizing that the masses are identical

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$v_{1i} + v_{2i} = v_{1f} + (1.5 + v_{1f}) = 2v_{1f} + 1.5$$

$$+2.0 - 0.5 = 2v_{1f} + 2.5$$

$$v_{1f} = -0.5 \text{ m/s} \quad ; \quad v_{2f} = 2.5 - 0.5 = 2.0 \text{ m/s}$$

> *In equal mass elastic collisions in one dimension, the masses simply exchange velocities. In equal mass, one dimensional elastic collisions with the first particle at rest, the second particle stops and the first particle goes forward with the original velocity of the second particle.*

There are a number of useful equations involving elastic collisions in special situations in one dimension. You don't have to memorize these, but they could be useful in solving certain problems. Also, study the examples of two–dimensional collisions.

Two Dimensional Collisions

Collisions between objects can also occur in two dimensions. The easiest example is two particles, m_1 and m_2 initially traveling towards each other on a straight line with velocities $\vec{\mathbf{v}}_{i1}$ and $\vec{\mathbf{v}}_{i2}$ This direction is conventionally called the x direction. The particles hit and then go off with velocity components in both the x and the y directions given by velocities $\vec{\mathbf{v}}_{1f}$ and $\vec{\mathbf{v}}_{2f}$ The basic principle in two dimensional collisions is the same as in one dimensional collisions:

Initial Momentum = Final Momentum

$$m_1\vec{\mathbf{v}}_{i1} + m_2\vec{\mathbf{v}}_{i2} = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f}$$

Elastic Two Dimensional Collisions

A typical two-dimensional collision involves particle m_1 traveling at known speed v_1 hitting particle m_2 which is initially at rest. After the collision particle m_1 goes off at velocity $\vec{\mathbf{v}}_{1f}$ which is at an angle θ with respect to (above) the original x axis. Particle m_2 goes off at velocity $\vec{\mathbf{v}}_{2f}$ which is at an angle ϕ with respect to the (below) original x axis.

We can now write the conservation of momentum equation as follows:

$$\text{X component} \quad m_1 v_{i1} = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi$$

$$\text{Y component} \quad 0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\phi$$

If the initial speed v_{i1} is known, then there are four unknowns in the right hand side: v_{1f}, v_{2f}, θ, and ϕ. Equivalently, each final velocity vector has two components, so that means four unknowns.

Since we have only two equations, this means that there is no unique answer available. So we need more equations ! If the collision is elastic, then we know that the kinetic energy before is equal to the kinetic energy after the collision

$$\frac{1}{2}m_1 v_{i1}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

So that means we can solve the problem if we specify one more of the unknowns.

Example of Elastic Two Dimensional Collisions

As an example of a two dimensional collision) we consider one proton traveling at known speed (3.5×10^5 m/s) colliding elastically with a second proton initially at rest. After the collision, one proton moves off at an angle θ of $37^{\text{O}}$with respect to the initial direction. What are the speeds of the two protons after the collision and what is the angle of the velocity of the second proton after the collision?
The solution is to write down the conservation of momentum along the two coordinate axes, and to use conservation of kinetic energy

$$\text{X component} \quad m_1 v_{i1} = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi$$

$$\text{Y component} \quad 0 = m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi$$

$$\frac{1}{2} m_1 v_{i1}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Since the masses are all the same, we can divide those out right away. We get

$$\text{X component} \quad 3.5 \times 10^5 = v_{1f} \cos 37 + v_{2f} \cos\phi$$

$$\text{Y component} \quad 0 = v_{1f} \sin 37 - v_{2f} \sin\phi$$

$$(3.5 \times 10^5)^2 = v_{1f}^2 + v_{2f}^2$$

There are three equations in three unknowns, so an exact solution is possible. It's actually very tedious to solve algebraically on paper. It's much easier to program it up on a spreadsheet or on a computer, and solve it that way. The result in this case is

$$v_{1f} = 2.80 \times 10^5 \text{ m/s}$$

$$v_{2f} = 2.11 \times 10^5 \text{ m/s}$$

$$\phi = 53.0^{\text{O}}$$

Notice that $\theta + \phi = 37 + 53 = 90^{\text{O}}$. This is always true when equal masses collide elastically.

The Center-of-Mass

Special Property of the Center-of-Mass

Consider a collection of say N particles with individual masses m_i where i ranges from 1 to N. Each of these particles has a position coordinate $\vec{r}_i$ with respect to a common Cartesian reference frame. Now assume that the particles are free to collide elastically and inelastically with one another, but *there are no external forces acting*. The resulting movement of the particles after the collisions may look to be very complicated. However, there is one simplifying feature. Namely *there exists a point called the center-of-mass whose velocity never changes.* If this center-of-mass point was not moving initially before any of the collisions take place, then it will not have moved after the collisions have taken place. Similarly, if the center-of-mass point was moving at constant velocity, it will continue to move at the same constant velocity before, during, and after the collisions. *In an isolated system which has no external forces acting, the center-of-mass has no acceleration!*

How to find the center-of-mass for N point particles

Finding the center-of-mass for N point particles is an easy mathematical exercise in Cartesian coordinates. We compute the averages of the x_i, y_i, and the z_i positions where those averages are weighted by the mass m_i of the particle at that position. If the M is the sum of all the m_i, then mathematically we have

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M} \; ; \; y_{CM} = \frac{\sum_{i=1}^{N} m_i y_i}{M} \; ; \; z_{CM} = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

$$\vec{r}_{CM} = x_{CM}\,\hat{\mathbf{i}} + y_{CM}\,\hat{\mathbf{j}} + z_{CM}\,\hat{\mathbf{k}} = \frac{\sum_{i=1}^{N} m_i x_i\,\hat{\mathbf{i}} + \sum_{i=1}^{N} m_i y_i\,\hat{\mathbf{j}} + \sum_{i=1}^{N} m_i z_i\,\hat{\mathbf{k}}}{M} = \frac{\sum_{i=1}^{N} mi\vec{r}_i}{M}$$

How to find the center-of-mass for an extended single mass

An extended (non-point) mass also has a center-of-mass point. If the mass is symmetrical, and of uniform density, then the center-of-mass point is at the geometric center of the shape. For an extended mass, the weight can be considered to be acting at the center-of-mass point. This has consequences, as in the *Leaning Tower of Pisa* class demonstration. The center-of-mass for an extended single mass is computed with integrals

$$x_{CM} = \frac{\int x\,dm}{M} \; ; \; y_{CM} = \frac{\int y\,dm}{M} \; ; \; z_{CM} = \frac{\int z\,dm}{M}$$

You must know the mass distribution to complete these integrations.

Motion of a System of Particles

The velocity of the center-of-mass for N particles

On the previous page we saw that the velocity of the center-of-mass for N particles is supposed to be constant in the absence of external forces. Here we can prove that statement.

First we obtain the formula for the center-of-mass velocity

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{v}_i$$

Now we cross multiply by the M to obtain

$$M\vec{v}_{CM} = \sum_{i=1}^{N} m_i \vec{v}_i = \sum_{i=1}^{N} \vec{p}_i = \vec{p}_{tot}$$

The above equation states that for a collection of N mass m_i, it appears that total momentum of the system is concentrated in a fictitious mass M moving with the velocity of the center-of-mass point. To the outside world, it's as if the N particles were all just one mass M located at the center-of-mass point.

The acceleration of the center-of-mass for N particles

Having obtained an expression for the velocity of the center-of-mass we can now look at the acceleration of the center-of-mass

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{a}_i$$

Again doing the multiplication by M we get the Newton's Second Law expression

$$M\vec{a}_{CM} = \sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_i = \vec{F}_{ext}$$

The fictitious total mass M is moving with an acceleration $\vec{a}_{CM}$ as given by a total external force. If the total external force is zero (only internal forces are acting) then the acceleration of the center-of-mass point is zero. This also means that the total momentum of the system, $\vec{p}_{tot}$ is constant.

CHAPTER 9: Rotation of a Rigid Body about a Fixed Axis

Up until know we have always been looking at "point particles" or the motion of the center–of–mass of extended objects. In this chapter we begin the study of *rotations* of an extended object about a fixed axis. Such objects are called **rigid bodies** because when they rotate they maintain their overall shape. It is just their orientation in space which is changing.

Angular Variables: θ, ω, α

The variables used to described the motion of "point particles" are displacement, velocity, and acceleration. For a rotating rigid body, there are three completely analogous variables: *angular displacement*, *angular velocity*, and *angular acceleration*. These angular variables are very useful because the they can be assigned to every point on the rigid body as it rotates about a fixed axis. The ordinary displacement, velocity, and acceleration can be calculated at a given point from the angular displacement, angular velocity, and angular acceleration just by multiplying by the distance r of that point from the axis of rotation.

Equations of motion with constant angular acceleration α:

For every equation which you have learned to describe the linear motion of a point particle, there is an exactly analogous equation to describe the rotational motion ($\theta(t)$ and $\omega(t)$) about a fixed axis.

Position with time: $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$ corresponds to $\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

Speed with time: $v(t) = v_0 + at$ corresponds to $\omega(t) = \omega_0 + \alpha t$

Speed with distance: $v^2(x) = v_0^2 + 2a(x - x_0)$ corresponds to $\omega^2(\theta) = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Relating linear kinematics with angular kinematics

For a purely rotating body, all points on the body move in circles about the axis of rotation. Therefore, we can relate the linear distance s moved by a point on the body to the angular displacement θ (*in radians!*) by knowing the radial distance r of that point from the axis of rotation. We have

$$s = r\theta \qquad 9.1$$

assuming that we defined the initial angular position of the point to be $\theta_0 = 0$

Quantitative Aspects of Rotational Motion

Relating the linear displacement s to the angular displacement θ

If a point P is on a rigid body, and that rigid body is rotated about some axis O which is a distance r away from the point P (see Fig. 9.2, page 286), then the point P will move a distance s given by

$$s = r\theta \tag{9.1}$$

where the angle θ is in radians. There are 2π radians in a full circle ($= 360^{\mathrm{O}}$), which makes one radian $\approx 57.3^{\mathrm{O}}$

Relating the linear speed v to the angular velocity ω

Take the same rotation as described above, and now add that the rotation is small amount $\Delta\theta$, and that it takes place in a small time interval Δt. We can now define the instantaneous angular velocity ω to be

$$\omega(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{9.2}$$

Since $v = ds/dt$ and $s = r\theta$, then we will have

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega(t) \tag{9.13}$$

The definition of a rigid body means that the distance r of the point P away from the axis of rotation does not change with time, so r can be treated as a constant in the derivation of this last equation.

Relating the linear acceleration a to the angular acceleration α

Finally, we consider the same description again of point P rotating, and now look at how fast the angular velocity ω is changing. Just as in linear motion, we define the linear acceleration to be the rate of change of the velocity variable, in rotational motion we define the angular acceleration to be the rate of change of the angular velocity

$$\alpha(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{9.6}$$

Don't forget that acceleration in two dimensions has two, perpendicular components: $\mathbf{a}_{total} = \mathbf{a}_{centripetal} + \mathbf{a}_{tangential}$, and these are given by:

$$a_{tangential} = r\alpha \qquad a_{centripetal} = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \tag{9.15}$$

Rotational Kinetic Energy

Consider a rigid body rotating with an angular velocity ω about an axis. Clearly every point in the rigid body (except where the axis is located) is moving at some speed v depending upon the distance away from the axis. Hence, the a rigid body in rotation must possess a kinetic energy. The formula for the rotational kinetic energy, you will see, is very similar to the formula for the kinetic energy of a moving point particle once you make the "translation" to the variables describing rotational motion.

In order to derive this formula, consider the rigid body to be composed of discrete mass elements m_i, where the distance of the mass element from the rotation axis is given as r_i. Then, from the previous notes, you know that the velocity of each mass element is given by $v_i = r_i\omega$. Note that each mass element can have a different amount of mass m_i, and can be at different distance r_i, but all the mass elements have a common angular velocity ω. The kinetic energy of the rigid body is then simply the sum of the individual kinetic energies of all the mass elements m_i:

$$K_{rotational} = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I\omega^2 \tag{9.17}$$

Here we introduce the **moment of inertia** I of the rigid body which is defined as summation quantity inside the parenthesis:

$$I \equiv \sum_i m_i r_i^2 \tag{9.16}$$

The Moment of Inertia I

The moment of inertia I is the rotational analog of the mass of a point particle. The moment of inertia depends not only on how much mass there is, but also on where that mass is located with respect to the rotation axis. So the shape of the rigid body must be specified, as well as the location of the rotation axis before the moment of inertia can be calculated. For an arbitrarily shaped rigid body having a density ρ, then the moment of inertia has to be calculated as an integral. This has been done for many common shapes (see Table 9.2, page 299)

$$I = \lim_{\Delta m_i \to 0}\left(\sum_i \Delta m_i r_i^2\right) = \int r^2 dm = \int \rho^2 dV$$