Chapter 8: The Center-of-Mass

Special Property of the Center-of-Mass

Consider a collection of say N particles with individual masses m_i where i ranges from 1 to N. Each of these particles has a position coordinate $\vec{r}_i$ with respect to a common Cartesian reference frame. Now assume that the particles are free to collide elastically and inelastically with one another, but *there are no external forces acting.* The resulting movement of the particles after the collisions may look to be very complicated. However, there is one simplifying feature. Namely *there exists a point called the center-of-mass whose velocity never changes.* If this center-of-mass point was not moving initially before any of the collisions take place, then it will not have moved after the collisions have taken place. Similarly, if the center-of-mass point was moving at constant velocity, it will continue to move at the same constant velocity before, during, and after the collisions. *In an isolated system which has no external forces acting, the center-of-mass has no acceleration!*

How to find the center-of-mass for N point particles

Finding the center-of-mass for N point particles is an easy mathematical exercise in Cartesian coordinates. We compute the averages of the x_i, y_i, and the z_i positions where those averages are weighted by the mass m_i of the particle at that position. If the M is the sum of all the m_i, then mathematically we have

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M} \; ; \; y_{CM} = \frac{\sum_{i=1}^{N} m_i y_i}{M} \; ; \; z_{CM} = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

$$\vec{r}_{CM} = x_{CM}\,\hat{\mathbf{i}} + y_{CM}\,\hat{\mathbf{j}} + z_{CM}\,\hat{\mathbf{k}} = \frac{\sum_{i=1}^{N} m_i x_i\,\hat{\mathbf{i}} + \sum_{i=1}^{N} m_i y_i\,\hat{\mathbf{j}} + \sum_{i=1}^{N} m_i z_i\,\hat{\mathbf{k}}}{M} = \frac{\sum_{i=1}^{N} mi\vec{r}_i}{M}$$

How to find the center-of-mass for an extended single mass

An extended (non-point) mass also has a center-of-mass point. If the mass is symmetrical, and of uniform density, then the center-of-mass point is at the geometric center of the shape. For an extended mass, the weight can be considered to be acting at the center-of-mass point. This has consequences, as in the *Leaning Tower of Pisa* class demonstration. The center-of-mass for an extended single mass is computed with integrals

$$x_{CM} = \frac{\int x\,dm}{M} \; ; \; y_{CM} = \frac{\int y\,dm}{M} \; ; \; z_{CM} = \frac{\int z\,dm}{M}$$

You must know the mass distribution to complete these integrations.

Motion of a System of Particles

The velocity of the center-of-mass for N particles

On the previous page we saw that the velocity of the center-of-mass for N particles is supposed to be constant in the absence of external forces. Here we can prove that statement.

First we obtain the formula for the center-of-mass velocity

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{v}_i$$

Now we cross multiply by the M to obtain

$$M\vec{v}_{CM} = \sum_{i=1}^{N} m_i \vec{v}_i = \sum_{i=1}^{N} \vec{p}_i = \vec{p}_{tot}$$

The above equation states that for a collection of N mass m_i, it appears that total momentum of the system is concentrated in a fictitious mass M moving with the velocity of the center-of-mass point. To the outside world, it's as if the N particles were all just one mass M located at the center-of-mass point.

The acceleration of the center-of-mass for N particles

Having obtained an expression for the velocity of the center-of-mass we can now look at the acceleration of the center-of-mass

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{a}_i$$

Again doing the multiplication by M we get the Newton's Second Law expression

$$M\vec{a}_{CM} = \sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_i = \vec{F}_{ext}$$

The fictitious total mass M is moving with an acceleration $\vec{a}_{CM}$ as given by a total external force. If the total external force is zero (only internal forces are acting) then the acceleration of the center-of-mass point is zero. This also means that the total momentum of the system, $\vec{p}_{tot}$ is constant.

CHAPTER 9: Rotation of a Rigid Body about a Fixed Axis

Up until know we have always been looking at "point particles" or the motion of the center–of–mass of extended objects. In this chapter we begin the study of *rotations* of an extended object about a fixed axis. Such objects are called **rigid bodies** because when they rotate they maintain their overall shape. It is just their orientation in space which is changing.

Angular Variables: θ, ω, α

The variables used to described the motion of "point particles" are displacement, velocity, and acceleration. For a rotating rigid body, there are three completely analogous variables: *angular displacement*, *angular velocity*, and *angular acceleration*. These angular variables are very useful because the they can be assigned to every point on the rigid body as it rotates about a fixed axis. The ordinary displacement, velocity, and acceleration can be calculated at a given point from the angular displacement, angular velocity, and angular acceleration just by multiplying by the distance r of that point from the axis of rotation.

Equations of motion with constant angular acceleration α:

For every equation which you have learned to describe the linear motion of a point particle, there is an exactly analogous equation to describe the rotational motion ($\theta(t)$ and $\omega(t)$) about a fixed axis.

Position with time: $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$ corresponds to $\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

Speed with time: $v(t) = v_0 + at$ corresponds to $\omega(t) = \omega_0 + \alpha t$

Speed with distance: $v^2(x) = v_0^2 + 2a(x - x_0)$ corresponds to $\omega^2(\theta) = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Relating linear kinematics with angular kinematics

For a purely rotating body, all points on the body move in circles about the axis of rotation. Therefore, we can relate the linear distance s moved by a point on the body to the angular displacement θ (*in radians!*) by knowing the radial distance r of that point from the axis of rotation. We have

$$s = r\theta \qquad 9.1$$

assuming that we defined the initial angular position of the point to be $\theta_0 = 0$

Quantitative Aspects of Rotational Motion

Relating the linear displacement s to the angular displacement θ

If a point P is on a rigid body, and that rigid body is rotated about some axis O which is a distance r away from the point P (see Fig. 9.2, page 286), then the point P will move a distance s given by

$$s = r\theta \tag{9.1}$$

where the angle θ is in radians. There are 2π radians in a full circle ($= 360^{\rm O}$), which makes one radian $\approx 57.3^{\rm O}$

Relating the linear speed v to the angular velocity ω

Take the same rotation as described above, and now add that the rotation is small amount $\Delta\theta$, and that it takes place in a small time interval Δt. We can now define the instantaneous angular velocity ω to be

$$\omega(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{9.2}$$

Since $v = ds/dt$ and $s = r\theta$, then we will have

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega(t) \tag{9.13}$$

The definition of a rigid body means that the distance r of the point P away from the axis of rotation does not change with time, so r can be treated as a constant in the derivation of this last equation.

Relating the linear acceleration a to the angular acceleration α

Finally, we consider the same description again of point P rotating, and now look at how fast the angular velocity ω is changing. Just as in linear motion, we define the linear acceleration to be the rate of change of the velocity variable, in rotational motion we define the angular acceleration to be the rate of change of the angular velocity

$$\alpha(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{9.6}$$

Don't forget that acceleration in two dimensions has two, perpendicular components: $\mathbf{a}_{total} = \mathbf{a}_{centripetal} + \mathbf{a}_{tangential}$, and these are given by:

$$a_{tangential} = r\alpha \qquad a_{centripetal} = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \tag{9.15}$$

Rotational Kinetic Energy

Consider a rigid body rotating with an angular velocity ω about an axis. Clearly every point in the rigid body (except where the axis is located) is moving at some speed v depending upon the distance away from the axis. Hence, the a rigid body in rotation must possess a kinetic energy. The formula for the rotational kinetic energy, you will see, is very similar to the formula for the kinetic energy of a moving point particle once you make the "translation" to the variables describing rotational motion.

In order to derive this formula, consider the rigid body to be composed of discrete mass elements m_i, where the distance of the mass element from the rotation axis is given as r_i. Then, from the previous notes, you know that the velocity of each mass element is given by $v_i = r_i\omega$. Note that each mass element can have a different amount of mass m_i, and can be at different distance r_i, but all the mass elements have a common angular velocity ω. The kinetic energy of the rigid body is then simply the sum of the individual kinetic energies of all the mass elements m_i:

$$K_{rotational} = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2}\Big(\sum_i m_i r_i^2\Big)\omega^2 = \frac{1}{2} I \omega^2 \tag{9.17}$$

Here we introduce the **moment of inertia** I of the rigid body which is defined as summation quantity inside the parenthesis:

$$I \equiv \sum_i m_i r_i^2 \tag{9.16}$$

The Moment of Inertia I

The moment of inertia I is the rotational analog of the mass of a point particle. The moment of inertia depends not only on how much mass there is, but also on where that mass is located with respect to the rotation axis. So the shape of the rigid body must be specified, as well as the location of the rotation axis before the moment of inertia can be calculated. For an arbitrarily shaped rigid body having a density ρ, then the moment of inertia has to be calculated as an integral. This has been done for many common shapes (see Table 9.2, page 299)

$$I = \lim_{\Delta m_i \to 0}\Big(\sum_i \Delta m_i r_i^2\Big) = \int r^2 dm = \int r^2 \rho dV$$

Chapter 10: Torque and Angular Acceleration

So far we have seen the rotational analogs of displacement, velocity, angular acceleration, mass, and kinetic energy. The last analog variable to be considered in this chapter is **torque** which is comparable to force. Note that all of these "analogous variables" in rotational motion have different units than the linear motion variables, and the same is true of torque. It is related to, but not the same thing, as force.

We introduced force (in Newton's second law $\mathbf{F} = \mathbf{ma}$) as the quantity which causes a mass to accelerate. We can do the same thing with the concept of **torque**. A torque, τ, is the quantity which causes a rigid body to undergo an angular acceleration with respect to some axis of rotation. In fact we have a very similar formula relating torque and angular acceleration:

$$\tau = I\alpha \tag{10.7}$$

This is closely related to Newton's second law where on the right hand side we are using the rotational motion variables moment of inertia and angular acceleration, instead of the linear motion variables mass and linear acceleration.

Torque: a qualitative description

The quantity torque is that which causes a rigid body to have a rotational acceleration about some axis. In order to give a rigid body a rotational acceleration, it is clear that one has to exert a force. However, *where the force is applied makes a difference.* If applies a force whose line of action goes through the proposed axis of rotation, then no rotation will occur. All that will happen is that the axis of rotation will exert an oppositely directed force and no motion will occur. The best example is that of a door where the axis of rotation is the door hinges. If you exert a force on the door close to or right at axis of the door hinges, then you will have a very difficult time opening a door. Instead you exert the force at the farthest possible distance from the door hinge, and perpendicular ($\theta = 90^{\mathrm{O}}$) to the distance from the axis. This provides you with the maximum *torque* for a given about of force. The general formula for torque is

$$\tau = Fr\sin\theta \tag{10.3}$$

The angle θ is the angle between the line of action of the force F and the distance r from the axis of rotation.

Worked Example in Rotational Acceleration

A solid cylinder of outer radius R_1 has an inner axis of radius R_2 ($R_2 < R_1$) through its center. A rope wrapped clockwise around R_1 exerts a force F_1, while a second rope wrapped around the radius R_2 in the opposite direction exerts a force F_2. Both forces are exerted perpendicular to the radius vector from the axis center. What is the net torque exerted on the cylinder?

The force F_1 tends to turn the cylinder in a clockwise direction. By convention, torques which cause *clockwise* acceleration have a *negative sign*:

$$\tau_1 = -F_1 \cdot R_1 \cdot \sin 90^{\mathrm{O}} = -F_1 R_1$$

The force F_2 tends to turn the cylinder in a counterclockwise direction. Again, by convention, torques which cause *counterclockwise* acceleration have a *positive sign*:

$$\tau_2 = +F_2 \cdot R_2 \cdot \sin 90^{\mathrm{O}} = +F_2 R_2$$

Then total torque on the cylinder is the sum of τ_1 and τ_2

$$\tau_{net} = \tau_1 + \tau_2 = -F_1 R_1 + +F_2 R_2$$

To give a specific case, suppose $F_1 = 5$ N, $F_2 = 6$ N, $R_1 = 1.0$ m, and $R_2 = 0.5$ m. The net torque is then

$$\tau_{net} = -F_1 R_1 + +F_2 R_2 = -5 \cdot 1.0 + 6 \cdot 0.5 = -2 \text{ N-m}$$

Since the net torque is negative, this means that the cylinder will rotate in the clockwise direction.

Note that the units of torque are Newton–meters (N–m). So don't confuse torque with force; they are different quantities. (You have previously learned another quantity with units of N–m. Do you recall that quantity?)

Torque and Angular Momentum

Worked Example

A uniform rod of length L and mass M is free to rotate about a pivot at the left end of the rod. The rod is initially in a horizontal position, and then is released. What is the *initial* angular acceleration of the right end of the rod, and the *initial* linear acceleration?

From the description of the problem, you should quickly see that the rod will just swing down (clockwise) much like a pendulum. Physically what is happening is that the at the center–of–mass of the rod, the force of gravity is exerting a torque with respect to the pivot point

$$\tau_{gravity} = (Mg)\frac{L}{2}\sin 90^{\mathrm{O}} = \frac{MgL}{2}$$

(This actually should have a negative sign since it is a clockwise torque, but since we are not worried about balancing other torques, then the sign is ignored.)

Now the "Newton's Second Law of Rotational Motion" relates the torque to the angular acceleration by using the moment of inertia I

$$\tau = I_{rod}\alpha$$

$$I_{rod} = \frac{1}{3}ML^2 \quad \text{(thin rod about axis at one end)}$$

$$\tau = I_{rod}\alpha = \frac{1}{3}ML^2\alpha = \frac{MgL}{2} \Longrightarrow \alpha = \frac{3g}{2L}$$

This angular acceleration is common to all points along the rod. To get the tangential acceleration at the end of the rod, you have to multiply α by the distance of the end of the rod from the pivot point

$$a_{tangential} = L\alpha = \frac{3}{2}g$$

Believe or NOT: **This value of acceleration is actually greater than g !**

Finally, does the pivot point exert any force on the rod? Think of the situation at the end when the rod has come down to a vertical position. If the pivot point is exerting a force, why don't we use it too when computing the net torque?

Worked Example
A wheel of radius R, mass M, and moment of inertia I is mounted on a horizontal axle. A mass m is vertically attached by a light cord wrapped around the circumference of the wheel. The m is dropping, and the wheel is rotating, both with an acceleration. Calculate the angular acceleration of the wheel, the linear acceleration of the mass m, and the tension in the cord.
This is a good example with which to test your comprehension of torques and rotational motion. You really should understand this solution thoroughly before being satisfied that you know about rotational motion and torques.
The solution to this problem just requires the use of Newton's second law both in its linear and rotational forms. Three equations will be produced. First, for the mass m, the net force acting on m is

$$F_m = mg - T = ma_m$$

where T is the tension in the cord supporting the mass.
Next, for the wheel, the tension T acts to produce a torque with respect to the axis of rotation. This torque is given by

$$\tau_T = TR = I\alpha = \frac{1}{2}MR^2\alpha$$

where we have used the expression for I valid for a solid disk.
Finally, because the cord is inextensible, the linear acceleration of the mass m is communicated to the tangential acceleration of the wheel, which in turn is related to the angular acceleration of the wheel

$$a_m = a_{tangential} = R\alpha$$

Now work backwards substituting first for α, and then for T

$$\alpha = \frac{a_m}{R} \Longrightarrow TR = \frac{1}{2}MR^2\frac{a_m}{R} \Longrightarrow T = \frac{Ma_m}{2}$$

Now substitute in the F_m equation

$$mg - T = ma_m \Longrightarrow mg - \frac{Ma_m}{2} = ma_m$$

$$\Longrightarrow a_m = \frac{2gm}{M + 2m} \quad ; \quad \alpha = \frac{2gm}{R(2m + M)} \quad ; \quad T = \frac{Mmg}{2m + M}$$

Angular Momentum of Rigid Bodies and Single Particles

We define the *angular momentum of a rigid body* rotating about an axis is

$$L = I\omega$$

Angular momentum is a vector. For simplicity we deal with symmetric rigid bodies rotating about one of their symmetry axes. For these cases, the direction of the angular momentum is given by the right hand rule. Curl the fingers of your right hand in the direction that the rigid body is rotating. Your thumb will point in the direction of the angular momentum.

The angular momentum of a point particle

The basic definition of angular momentum is for a point particle moving at some velocity **v** and at a vector distance **r** away from some reference axis. The angular momentum of the point particle is given by

$$\vec{\mathbf{l}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = m(\vec{\mathbf{r}} \times \vec{\mathbf{v}})$$

Conservation of Angular Momentum of a Rigid Body

You remember that we can write Newton's Second law as force is the time rate of change of linear momentum

$$\vec{F} = \frac{d\vec{p}}{dt}$$

We have the equivalent to Newton's Second Law for rotations

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

The **net toque** is equal to the time rate of change of **angular momentum**. Now when there is no external torque, then the angular momentum of a rigid body must remain constant. A great example of angular momentum the spinning of an ice skater. When there is no external torque angular momentum must be conserved. We write

$$L_i = L_f$$