Chapter 10: Torque and Angular Acceleration

So far we have seen the rotational analogs of displacement, velocity, angular acceleration, mass, and kinetic energy. The last analog variable to be considered in this chapter is **torque** which is comparable to force. Note that all of these "analogous variables" in rotational motion have different units than the linear motion variables, and the same is true of torque. It is related to, but not the same thing, as force.

We introduced force (in Newton's second law $\mathbf{F} = \mathbf{ma}$) as the quantity which causes a mass to accelerate. We can do the same thing with the concept of **torque**. A torque, τ, is the quantity which causes a rigid body to undergo an angular acceleration with respect to some axis of rotation. In fact we have a very similar formula relating torque and angular acceleration:

$$\tau = I\alpha \tag{10.7}$$

This is closely related to Newton's second law where on the right hand side we are using the rotational motion variables moment of inertia and angular acceleration, instead of the linear motion variables mass and linear acceleration.

Torque: a qualitative description

The quantity torque is that which causes a rigid body to have a rotational acceleration about some axis. In order to give a rigid body a rotational acceleration, it is clear that one has to exert a force. However, *where the force is applied makes a difference.* If applies a force whose line of action goes through the proposed axis of rotation, then no rotation will occur. All that will happen is that the axis of rotation will exert an oppositely directed force and no motion will occur. The best example is that of a door where the axis of rotation is the door hinges. If you exert a force on the door close to or right at axis of the door hinges, then you will have a very difficult time opening a door. Instead you exert the force at the farthest possible distance from the door hinge, and perpendicular ($\theta = 90^{\mathrm{O}}$) to the distance from the axis. This provides you with the maximum *torque* for a given about of force. The general formula for torque is

$$\text{Vector Definition: } \vec{\tau} = \vec{r} \times \vec{F} \Longrightarrow \text{ Magnitude: } \tau = rF\sin\theta \tag{10.3}$$

The angle θ is the angle between the line of action of the force F and the distance r from the axis of rotation. The angle of the vector $\vec{\tau}$ is according to the *right-hand rule*.

Worked Example in Rotational Acceleration

A solid cylinder of outer radius R_1 has an inner axis of radius R_2 ($R_2 < R_1$) through its center. A rope wrapped clockwise around R_1 exerts a force F_1, while a second rope wrapped around the radius R_2 in the opposite direction exerts a force F_2. Both forces are exerted perpendicular to the radius vector from the axis center. What is the net torque exerted on the cylinder?
The force F_1 tends to turn the cylinder in a clockwise direction. By convention, torques which cause *clockwise* acceleration have a *negative sign*:

$$\tau_1 = -F_1 \cdot R_1 \cdot \sin 90^{\mathrm{O}} = -F_1 R_1$$

The force F_2 tends to turn the cylinder in a counterclockwise direction. Again, by convention, torques which cause *counterclockwise* acceleration have a *positive sign*:

$$\tau_2 = +F_2 \cdot R_2 \cdot \sin 90^{\mathrm{O}} = +F_2 R_2$$

Then total torque on the cylinder is the sum of τ_1 and τ_2

$$\tau_{net} = \tau_1 + \tau_2 = -F_1 R_1 + +F_2 R_2$$

To give a specific case, suppose $F_1 = 5$ N, $F_2 = 6$ N, $R_1 = 1.0$ m, and $R_2 = 0.5$ m. The net torque is then

$$\tau_{net} = -F_1 R_1 + +F_2 R_2 = -5 \cdot 1.0 + 6 \cdot 0.5 = -2 \text{ N-m}$$

Since the net torque is negative, this means that the cylinder will rotate in the clockwise direction.
Note that the units of torque are Newton–meters (N–m). So don't confuse torque with force; they are different quantities. (You have previously learned another quantity with units of N–m. Do you recall that quantity?)

Things to Know about Cross-Products

From the general definition of the cross-product of two vectors, you should be able to prove the following

$$\text{For any two vectors} \quad \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \text{and} \quad \vec{A} \times \vec{A} = 0$$

$$\text{For the units vectors} \quad \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \quad ; \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \quad ; \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

Torque and Angular Momentum

Worked Example using Torques

A uniform rod of length L and mass M is free to rotate about a pivot at the left end of the rod. The rod is initially in a horizontal position, and then is released. What is the *initial* angular acceleration of the right end of the rod, and the *initial* linear acceleration?

From the description of the problem, you should quickly see that the rod will just swing down (clockwise) much like a pendulum. Physically what is happening is that the at the center–of–mass of the rod, the force of gravity is exerting a torque with respect to the pivot point

$$\tau_{gravity} = (Mg)\frac{L}{2}\sin 90^{\mathrm{O}} = \frac{MgL}{2}$$

(This actually should have a negative sign since it is a clockwise torque, but since we are not worried about balancing other torques, then the sign is ignored.)

Now the "Newton's Second Law of Rotational Motion" relates the torque to the angular acceleration by using the moment of inertia I

$$\tau = I_{rod}\alpha$$

$$I_{rod} = \frac{1}{3}ML^2 \quad \text{(thin rod about axis at one end)}$$

$$\tau = I_{rod}\alpha = \frac{1}{3}ML^2\alpha = \frac{MgL}{2} \Longrightarrow \alpha = \frac{3g}{2L}$$

This angular acceleration is common to all points along the rod. To get the tangential acceleration at the end of the rod, you have to multiply α by the distance of the end of the rod from the pivot point

$$a_{tangential} = L\alpha = \frac{3}{2}g$$

Believe or NOT: **This value of acceleration is actually greater than g !**

Finally, does the pivot point exert any force on the rod? Think of the situation at the end when the rod has come down to a vertical position. If the pivot point is exerting a force, why don't we use it too when computing the net torque?

Worked Example using Torques and Linear Motion
A wheel of radius R, mass M, and moment of inertia I is mounted on a horizontal axle. A mass m is vertically attached by a light cord wrapped around the circumference of the wheel. The m is dropping, and the wheel is rotating, both with an acceleration. Calculate the angular acceleration of the wheel, the linear acceleration of the mass m, and the tension in the cord.
This is a good example with which to test your comprehension of torques and rotational motion. You really should understand this solution thoroughly before being satisfied that you know about rotational motion and torques.
The solution to this problem just requires the use of Newton's second law both in its linear and rotational forms. Three equations will be produced. First, for the mass m, the net force acting on m is

$$F_m = mg - T = ma_m$$

where T is the tension in the cord supporting the mass.
Next, for the wheel, the tension T acts to produce a torque with respect to the axis of rotation. This torque is given by

$$\tau_T = TR = I\alpha = \frac{1}{2}MR^2\alpha$$

where we have used the expression for I valid for a solid disk.
Finally, because the cord is inextensible, the linear acceleration of the mass m is communicated to the tangential acceleration of the wheel, which in turn is related to the angular acceleration of the wheel

$$a_m = a_{tangential} = R\alpha$$

Now work backwards substituting first for α, and then for T

$$\alpha = \frac{a_m}{R} \Longrightarrow TR = \frac{1}{2}MR^2\frac{a_m}{R} \Longrightarrow T = \frac{Ma_m}{2}$$

Now substitute in the F_m equation

$$mg - T = ma_m \Longrightarrow mg - \frac{Ma_m}{2} = ma_m$$

$$\Longrightarrow a_m = \frac{2gm}{M + 2m} \;\; ; \;\; \alpha = \frac{2gm}{R(2m + M)} \;\; ; \;\; T = \frac{Mmg}{2m + M}$$

Angular Momentum of Rigid Bodies and Single Particles

We define the *angular momentum of a rigid body* rotating about an axis is

$$L = I\omega$$

Angular momentum is a vector. For simplicity we deal with symmetric rigid bodies rotating about one of their symmetry axes. For these cases, the direction of the angular momentum is given by the right hand rule. Curl the fingers of your right hand in the direction that the rigid body is rotating. Your thumb will point in the direction of the angular momentum.

The angular momentum of a point particle

The basic definition of angular momentum is for a point particle moving at some velocity **v** and at a vector distance **r** away from some reference axis. The angular momentum of the point particle is given by

$$\vec{\mathbf{l}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = m(\vec{\mathbf{r}} \times \vec{\mathbf{v}})$$

Conservation of Angular Momentum of a Rigid Body

You remember that we can write Newton's Second law as force is the time rate of change of linear momentum

$$\vec{F} = \frac{d\vec{p}}{dt}$$

We have the equivalent to Newton's Second Law for rotations

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

The **net toque** is equal to the time rate of change of **angular momentum**. Now when there is no external torque, then the angular momentum of a rigid body must remain constant. A great example of angular momentum the spinning of an ice skater. When there is no external torque angular momentum must be conserved. We write

$$L_i = L_f$$

Worked Example in Angular Momentum Conservation

Problem Statement, see pages 334–335

A professor has a moment of inertia $I_P = 3.0$ kg-m^2 about his symmetry axis, with his arms outstretched. The professor standing on a turntable, holding a dumbbell in each hand. The dumbbells weigh 5.0 kg each, and are at a distance of 1.0 m from the professor's axis of symmetry. The professor is set spinning at a rate of 1 revolution every 2 seconds.

The professor draws the dumbbells into his chest, effectively 0.20 m from the symmetry axis. His new moment of inertia becomes $I'_P = 2.2$ kg-m^2. His angular rotation speed is observed to increase. Why? What is his new angular rotation speed? How does the rotational kinetic energy compare before and after the professor has drawn in the two dumbbells?

Problem Solution

The professors angular velocity increases because of angular momentum conservation. This is the same effect as when an ice skater is spinning with his/her arms outstretched, and then draws the arms in closely. The dumbbells enhance the effect. Here is the numerical solution (page 335).

$$\text{Angular momentum before} \quad = \quad \text{Angular momentum after}$$

$$I_1\omega_1 = I_2\omega_2$$

The total moment of inertia before I_1 is the sum of the professor's moment of inertia before and the contribution of the dumbbells which are initially 1.0 m from the professor's symmetry axis:

$$I_1 = 3.0 + 2 * 5.0 * (1.0)^2 = 13 \text{ kg-m}^2$$

After the professor brings in the two dumbbells to 0.20 m distance, the new total moment of inertia I_2 is

$$I_2 = 2.2 + 2 * 5.0 * (0.2)^2 = 2.6 \text{ kg-m}^2$$

Clearly since $I_2 < I_1$, then in order to conserve angular momentum we must have $\omega_2 > \omega_1$. In fact

$$\omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{13.6}{2.6}(\frac{1 \text{ revolution}}{2 \text{ seconds}}) = 2.5 \text{ rev/sec}$$

The professors angular velocity increases by a factor of five, from 0.5 to 2.5 rev/sec.

Worked Example in Angular Momentum Conservation

Problem Solution

As for the kinetic energy, we have $K_1 = I_1\omega_1^2/2$, and $K_2 = I_2\omega_2^2/2$. We have to convert that ω from revolutions per second to radians per second, where one revolution is 2π radians. After we do that we will find that $K_1 = 64$ J and $K_2 = 320$ J. Why has the mechanical energy increased so much, the same factor of five in fact?

Chapter 11: Rotational Equilibrium

With the use of torques one can solve problems in **rotational equilibrium**. Rotational equilibrium is the rigid body equivalent of the equilibrium of a particle for which the net external force is 0.
Rotational equilibrium of a rigid body is summarized by the following statement:

The net clockwise torque equals the net counterclockwise torque

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Statement of Problem
As an example consider a uniform horizontal beam of length 8.00 m, and weight 200 N. The beam is free to rotate about a pivot in a wall on one end of the beam. The other end of the beam is tied to a cable making an angle of 53 degrees with respect to the beam. A 600 N man is standing 2.00 m away from the wall. What is the tension in the cable, and the force exerted by the pivot on the beam?
Method of Solution
Again this is an equilibrium problem. Nothing is being accelerated. So the net force must be zero in the vertical and the horizontal directions. However, there is more than that. There is no rotation either. So the clockwise torque must be equal to the counter clockwise torque. Essentially we have three equations: one for the vertical forces, one for the horizontal forces, and one for the torques. We have three unknowns: the horizontal force of the pivot, the vertical force of the pivot, and the tension in the cable. So we can solve for the three unknowns from the three equilibrium equations.

Worked Example in Rotational Equilibrium

Implementation of Solution

Call the pivot force $\vec{R}$, and assume that it acts at an angle θ with respect to the beam. Call the tension force in the cable $\vec{T}$, where we are told that $\vec{T}$ acts at an angle of 53 degrees. So we have the following two net force = 0 equilibrium equations

$$\sum F_x = 0 = R\cos\theta - T\cos 53$$
$$\sum F_y = 0 = R\sin\theta + T\sin 53 - 600 - 200$$

Our third equation is for the sum of the torques being 0. We can choose the axis of rotation any where we want. However, the smart thing to do is chose an axis corresponding to the location of one of the forces. That way, the torque from that force is automatically 0. The really smart thing to do is choose an axis where an unknown force is acting. Then you don't have to worry about that unknown force. In this case the really smart axis choice is at the pivot point. That give the net toque equation as

$$\sum \tau = 0 = (T\sin 53)(8.00) - (600)(2.00) - (200)(4.00)$$

Right away we can solve for $T = 313$ Newtons. We then plug this value of T into our force equations to get

$$R\cos\theta = 188 \text{ Newtons}$$

$$R\sin\theta = 550 \text{ Newtons}$$

We divide the second of these equations by the first to obtain $\tan\theta = 2.93 \Longrightarrow \theta = 71.1$ degrees. And with this value of θ we find $R = 581$ Newtons.