CHAPTER 13: Oscillatory Motion

Consider a spring lying in a horizontal position. A mass is attached to the spring, and the spring is stretched and then released. In the absence of friction the mass will *oscillate* along the horizontal axis. That means the position x of the mass will go from a maximum positive displacement, to zero displacement, and then to a maximum negative displacement, and then repeat the *cycle*. This position function $x(t)$ obeys the *Simple Harmonic Motion* equation (SHM):

$$\textbf{Simple Harmonic Motion} \quad x(t) = A\cos(\omega t + \phi) \tag{13.13}$$

The position function $x(t) = A\cos(\omega t + \phi)$ has three parameters:

$A \equiv$ is the **amplitude** of the motion

$\omega(= 2\pi f) \equiv$ is the **angular frequency** of the motion

$\phi \equiv$ is the **phase constant** of the motion

The time for one complete *cycle* of the oscillation is called the **period T**.
The number of cycles per second is called the **frequency f**. The frequency f is the inverse of the period T

$$\textbf{frequency} \quad f = 1/T$$

There are two simple systems, *pendulum* and *spring* for which you should now the equation for their periods in terms of the physical parameters:

Period for a Spring of a given mass and force constant: $T = 2\pi\sqrt{\dfrac{m}{k}}$

$$\textbf{Period for a Pendulum} \text{ of a given length: } T = 2\pi\sqrt{\frac{L}{g}} \tag{13.34}$$

The velocity and the acceleration of the mass in oscillatory motion can be calculated directly from the position function $x(t) = A\cos(\omega t + \phi)$ by taking the first and the second time derivatives respectively:

$$v(t) = \frac{dx}{dt} = \frac{d(A\cos(\omega t + \phi))}{dt} = -\omega A\sin(\omega t + \phi) \tag{13.15}$$

$$a(t) = \frac{dv}{dt} = \frac{d(-\omega A\sin(\omega t + \phi))}{dt} = -\omega^2 A\cos(\omega t + \phi) \tag{13.16}$$

Oscillatory or Simple Harmonic Motion

We will prove shortly that the general equation for simple harmonic motion is given by

$$x(t) \quad (\text{or } y(t)\ ) = A\cos\left(\omega t + \phi\right) \tag{13.13}$$

This equation describes the motion of a mass attached to a spring (either horizontally or vertically), or the motion of a pendulum.
The purpose of this chapter is to study this type of motion, and in particular to become familiar with the concepts of *Amplitude*, *Frequency*, and *Phase*.

Take a specific case of a spring which is stretched to an initial distance x_0, and an attached mass is also given an initial speed v_0. We will see that the constants A and ϕ can be expressed in terms of the two initial conditions x_0 and v_0.
We first substitute in the position and the velocity equations at time $t = 0$

$$x(t=0) = A\cos\left(\omega\cdot 0 + \phi\right) = A\cos\phi = x_0$$

$$v(t=0) = -\omega A\sin\left(\omega\cdot 0 + \phi\right) = -\omega A\sin\phi = v_0$$

Now divide the second equation by the first in order to get an equation for the phase angle by itself in terms of x_0 and v_0

$$\tan\phi = -\frac{v_0}{\omega x_0}$$

With a little more work, you can substitute this expression for $\tan\phi$ into one of the two other equations and obtain an expression for the amplitude A just in terms of x_0, v_0, and ω

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

Special case where $v_0 = 0$ (no initial velocity)

$$\tan\phi = 0 \Longrightarrow \phi = 0$$

$$A = \sqrt{x_0^2 + 0} \Longrightarrow A = x_0$$

So this special case gives a very simple Simple Harmonic Motion equation

$$x(t) = x_0\cos\omega t \qquad\qquad (\text{only when } v_0 = 0)$$

Worked example of Simple Harmonic Motion

A particle oscillates in simple harmonic motion according to the following position equation where t is in seconds and x is in meters:

$$x(t) = 4.0 \cos(\pi t + \frac{\pi}{4})$$

Here the *phase angle* ϕ is given in radians instead of the more common degrees. (Recall that π radians is equal to $180^{\rm O}$, so $\pi/4 = 45^{\rm O}$.)
Determine the amplitude, frequency (and angular frequency), and period of the motion. Determine the position, velocity, and acceleration of the particle at $t = 1$ second.

The amplitude A and the angular frequency ω can be determined by simply comparing this equation to the general equation $x(t) = A\cos(\omega t + \phi)$. You will see right away that in this example $A = 4.0$ meters, and $\omega = \pi$ radians/second. To get the (plain) frequency f, and then the period T requires that you recall the relation between the angular frequency ω and f

$$\omega = 2\pi f \Longrightarrow f = \frac{\omega}{2\pi} \tag{13.11}$$

So in this case $f = \pi/(2\pi) = 0.5$ cycles per second.
The period is given just as the inverse of the (plain) frequency f

$$T = \frac{1}{f} = \frac{1}{0.5\ \mathrm{s}^{-1}} = 2.0\ \mathrm{s}$$

To get the position and velocity at $t = 1$ second, just substitute in position and the velocity equations of motion:

$$x(t = 1) = 4.0 \cos(\pi \cdot 1 + \frac{\pi}{4}) = -2.83\ \mathrm{m}$$

$$v(t = 1) = -(4.0)(\pi)(\sin(\pi \cdot 1 + \frac{\pi}{4})) = 8.89\ \mathrm{m/s}$$

$$a(t = 1) = -(4.0)(\pi)^2(\cos(\pi \cdot 1 + \frac{\pi}{4})) = 27.9\ \mathrm{m/s}^2$$

Newton's Second Law and Simple Harmonic Motion

Up to now we have been working with the position equation

$$x(t) = A\cos(\omega t + \phi) \qquad (13.13)$$

for oscillatory motion, and have simply stated that this is the correct equation. Now we will prove that is the correct for the case of a mass attached to a spring. This we will do using Newton's second law of motion, the force equation
When a spring is stretched a distance x from its unstretched position where $x > 0$, then there will be a force in the negative x direction. Such a force is called a **Restoring Force** because it tends to restore the spring back to its original configuration. We have already seen in Chapter 6 that the magnitude of the force from a stretched or compressed spring depends upon the spring force constant k

$$F_{spring} = -kx \qquad (13.3)$$

The negative sign in this equation is the key feature of a Restoring Force. Such a force acts to return the spring to its unstretched length. This force will be exerted on a mass m which is attached to the spring. By Newton's second law we have

$$F = ma = -kx \Longrightarrow a = -\frac{k}{m}x \qquad (13.4)$$

Like the force itself, the acceleration is linearly proportional and opposite in sign to the displacement from the equilibrium position.
This equation can be re–written using the calculus definition of acceleration

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

We now symbolize the ratio k/m by the symbol ω^2, and then obtain:

$$\textbf{Angular Frequency:} \quad \omega \equiv \sqrt{\frac{k}{m}}$$

$$\Longrightarrow a = \frac{d^2x}{dt^2} = -\omega^2 x$$

The angular frequency $\omega = \sqrt{k/m}$ is an intrinsic characteristic of the mass-spring system. This means that the angular frequency for oscillation *does not* depend on any initial conditions of the motion.

Newton's Second Law and Simple Harmonic Motion

We have derived an equation for the position function $x(t)$:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

This equation states that we need a function $x(t)$ such that when we take two derivatives with respect to t, then we will get back the *negative* of the original function $x(t)$ multiplied by ω^2. Very simply, Eq. 13.3 is exactly that function, and the proof is simply just do it.
Start with Eq. 13.13, and then take two time derivatives:

$$x(t) = A\cos(\omega t + \phi) \tag{13.13}$$

$$\frac{d}{dt}x(t) = -A\omega\sin(\omega t + \phi)$$

$$\frac{d^2}{dt}x(t) = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x(t)$$

In general, whenever one has a restoring force equation coming in Newton's second law, then one will always find the solution to be the position function is just simple harmonic motion.
Note that the constants A and ϕ are completely arbitrary. Any pair on numbers A and ϕ will satisfy the restoring force equation. In order to fix A and ϕ to particular values, one must specify the initial conditions, namely the initial position (x_0), and the initial velocity (v_0) as we have done previously:

$$\tan\phi = -\frac{v_0}{\omega x_0}$$

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

REVIEW: The Mass–Spring System in Simple Harmonic Motion

The general solution for simple harmonic motion is

$$x(t) = A\cos(\omega t + \phi) \tag{13.13}$$

where for the spring we remember that

$$\omega^2 = \frac{k}{m} \Longrightarrow \omega = \sqrt{\frac{k}{m}} \tag{13.9}$$

Now in general $\omega = 2\pi f$ so

$$\omega = 2\pi f = \sqrt{\frac{k}{m}} \Longrightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{13.11}$$

Finally one can determine the period T of the motion in terms of k and m since $T = 1/f$

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \tag{13.12}$$

Worked Example

A car of mass 1300 kg has four shock absorber springs each with a force constant of 20,000 N/m. If two people riding in the car have a combined mass of 160 kg, what is the frequency f of the vibration of the car when it is driven over a pothole?

Assume that the total mass (=1460 kg) is equally distributed over the four shocks, so each shock absorber has a mass of 325 kg attached to it. The solution is just to use Eq. 13.11 to solve for f

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20,000}{365}} = 1.18 \text{ Hz}$$

The abbreviation Hz (after Heinrich Hertz) means 1 cycle/second. The period of the vibration T is simply given as the inverse of the frequency

$$T = \frac{1}{f} = \frac{1}{1.18} = 1.70 \text{ seconds}$$

Kinetic and Potential Energy in Simple Harmonic Motion

Kinetic Energy of a Mass in Oscillation

We have seen that the velocity in simple harmonic motion is continuously changing going from maximum to minimum to maximum and so on. Hence the kinetic energy must also be changing continuously

$$v(t) = \frac{d}{dt}x(t) = -x_m\omega\sin(\omega t + \phi)$$

Given the mass m, then the kinetic energy can always be calculated from the velocity v

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi)$$

Potential Energy of a Mass in Oscillation

Similarly the potential energy at any time may be calculated straight from the square of the position function, since for a spring the potential energy is given by $U(x) = (1/2)kx^2$

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}k\Big(x_m\cos(\omega t + \phi)\Big)^2 = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi)$$

Total Mechanical Energy of a Mass in Oscillation

Since we have assumed that there is no friction, then the total mechanical energy ($E_{total} = K + U$) must be a constant

$$E_{total} = K + U = \frac{1}{2}m\omega^2 x_m^2\sin^2(\omega t + \phi) + \frac{1}{2}kx_m^2\cos^2(\omega t + \phi)$$

Since $\sin^2\theta + \cos^2\theta = 1$, then we have simply

$$E_{total} = \frac{1}{2}kx_m^2$$

The total mechanical energy for a mass in simple harmonic motion is constant, and scales as the square of the amplitude of the motion.

In simple harmonic motion, the total energy is being exchanged continuously between the kinetic and the potential forms.

The Pendulum as Simple Harmonic Motion

Exact Differential Equation of Motion for a Pendulum

We have already seen the pendulum motion just in terms of the interchange of kinetic and potential energies in the previous chapters. Now we study the motion of the pendulum from the kinematics point of few. We want to find expression for the position and the velocity as functions of time.

Consider the mass m at the end of the string L to be displaced by the angle θ. There will be a net force $F_{tangential}$ on the mass perpendicular to the direction of the string. The force is just $mg\sin\theta$ (exactly like the weight force down the inclined plane). So we can write

$$F_{tangential} = -mg\sin\theta = ma_{tangential} = m\frac{d^2s}{dt^2}$$

Here s is the tangential displacement along a circular arc: $s = L\theta$ (where θ is in radians). So now Newton's second law is written as

$$F = m\frac{d^2s}{dt^2} = mL\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Approximate Differential Equation of Motion for a Pendulum

The above differential equation of motion is not quite a *restoring force* equation. However, we restrict ourselves to the cases where the angular displacement θ is small such that $\sin\theta \approx \theta$, and where θ is being expressed in radians (not degrees). In that case we have

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

This looks just like the spring equation, where now $\omega^2 = g/L$ instead of $\omega^2 = k/m$. By analogy then, one can write the period of a pendulum as

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad \text{(independent of the mass !)}$$

The Physical Pendulum

Besides the **Simple Pendulum** there is also the not–so–simple pendulum which is normally called the **Physical Pendulum**. This is an extended object (rigid body) instead of a point mass which is rotating because of a torque which occurs when the body is displaced from equilibrium. See Figure 13.23 on page 438.
The center-of-mass of the rigid body is at a distance d from a pivot point. The rigid body has a moment of inertia I about an axis through the pivot point. The line from the pivot point to the center-of-mass is at some angle θ with respect to the vertical direction. The weight vector of the rigid body, $m\vec{g}$ can be resolved into a component $mg\cos\theta$ along the line to the pivot point, and a perpendicular component $mg\sin\theta$. It is this perpendicular component which exerts the torque about the pivot point. The basic torque equation $\tau = I\alpha$ becomes (Fig. 13.23)

$$-(mg\sin\theta)d = I\frac{d^2\theta}{dt^2}$$

Essentially the length d in a physical pendulum replaces the string lenght l of the simple pendulum.
This torque is restoring (negative sign) because it tends to push the rigid body back to equilibrium, that is to reduce angular displacment. We again make the small angle approximation that $\sin\theta \approx \theta$ (in radians) to get

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta = -\omega^2\theta$$

We recognize this as another simple harmonic motion equation for which

$$\omega = \sqrt{\frac{mgd}{I}}$$

and the period is given as usual by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

The textbook has an interesting example of leg of a *Tyrannosaurus Rex* as a physical pendulum on pages 439–440. You can get an approximate value for the walking speed of a *T. Rex* as about 4 miles/hour, the same as a human being. Whether *T. Rex* could run as fast as a jeep, as in the movie *Jurassic Park*, is not discussed.

Chapter 14: Fluid Mechanics (Density, Pressure, and Pascal's Law)

Our study of Mechanics is almost complete. There remains just the subject of *Fluid Mechanics* which applies specifically to gases and liquids for which both the concepts of *pressure* and *density* make sense.
We have already seen in Chapter 1 that the density of an object (solid, liquid, or gas) is the ratio of the mass of the object divided by the volume the object occupies

$$\rho = \frac{m}{V} \tag{14.1}$$

For a tabulation of densities, see Table 14.1 on page 457.

The second quantity of importance in fluid mechanics is *pressure*. Pressure is the force per unit area that a gas or a liquid exerts on the surface of its confining volume. Very simply, pressure is the normal weight or perpendicular per unit area of a fluid or a gas

$$P \equiv \frac{F \perp}{A} \tag{14.3}$$

Pressure is measured in units of **Pascals** (Pa) where 1 Pa is 1 N/m^2. The most important feature about *Pressure* in a liquid or a gas is that the amount of the pressure varies with the depth of the liquid or the gas. The pressure difference between two different levels y_1 and y_2 of a liquid is given by

$$p_2 - p_1 = -\rho g(y_2 - y_1) \tag{14.5}$$

$$p(h) = p_{air} + \rho g h \quad \text{(Pressure at the bottom of a liquid open to air)} \tag{14.8}$$

An important consequence of these equations is that *the pressure is the same for all points of equal depth in any liquid.*

Pascal's Law states that a change in pressure on a fluid is transmitted equally to all points in the fluid and the enclosing surface of the container. This is the principle of the *hydraulic press.*

Variation of Pressure With Depth

Anyone who has dived into a swimming pool, or the ocean, knows that the pressure of the water increases with the depth below the surface. This is true of all liquids, and also of the atmosphere itself. The accumulated weight of the water (or the air) above a person is what is responsible for the pressure exerted at a certain depth. This can be deduced from just the **translational equilibrium** in the y direction. Consider a small, cubical element of fluid in a container, with a cross sectional area A, and a differential height dy. That cubical element has a volume $dV = Ady$, and contains a differential element of mass dm. That volume element is not moving, so:

$$\sum F_y = 0 = \text{Force acting up} - \text{Force acting down}$$

$$\sum F_y = 0 = (P + dP)A - dW - PA = AdP - g(dm) = AdP - g\rho A dy$$

$$\Longrightarrow \frac{dP}{dy} = +\rho g$$

The convention here is that the depth y is measured from the surface of the water, and y increases as you go deeper into the water. So the above equation says that the pressure increases with increasing depth. One can integrate the above derivative equation very simply to get

$$P(y) = \rho g y + \text{ constant}$$

This equations shows clearly that pressure is the same when y is the same. The constant of integration can be determined by knowing the pressure at $y = 0$. If the container is open to the atmosphere, then that pressure $P(y = 0) = P_a$ where P_a is the atmospheric pressure. This gives

$$P(y) = \rho g y + P_a \quad (P_a = 1.01 \text{ x } 10^5 \text{ Pa or } 14.7 \text{ lb/in}^2) \qquad (14.8)$$

The pressure at at depth y is equal to the atmospheric pressure plus an amount $\rho g y$ (the weight of a column of unit area with height y).

The Variation of Pressure with Depth

Example of a hydraulic lift

A hydraulic lift consists of a small diameter piston of radius 5 cm, and a large diameter piston of radius 15 cm. How much force must be exerted on the small diameter piston in order to support the weight of a car at 13,300 N ?

The pressure (F/A) on both sides of the hydraulic lift must be the same at the same height y. This lead to

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} \Longrightarrow F_1 = F_2\Big(\frac{A_1}{A_2}\Big)$$

$$F_1 = 13,300\Big(\frac{\pi(0.05)^2}{\pi(0.15)^2}\Big) = 1.48 \text{ x } 10^3 \text{ N}$$

There is a factor of 9 gain in lifting power by means of the hydraulic press. The same force multiplication occurs in the braking system of cars which use brake fluid to transmit the force from the brake pedal.

Second Example

Calculate the pressure at an ocean depth of 1000 m, using the density of water as 1.0 x 10^3 kg/m^3.

From Eq. 14.8 we have

$$P(y = 1000) = \rho g(1000) + 1.01 \text{ x } 10^5 = 9.90 \text{ x } 10^6 \text{ Pa}$$

This pressure is 100 times that of normal atmospheric pressure. Now you know why submarines don't have portholes.