Third Exam Chapter Study Guide

Chapter 9

1) Kinematic definitions for rotational variables θ, ω, and α; three kinematic equations relating rotational variables for constant acceleration
2) Constrained, no-slip linear motion and rotational motion
3) Computation of moment of inertia for discrete masses, and kinetic energy
4) Don't need to know sections 9.5 and 9.6 (parallel axis, calculus for I)

Chapter 10

1) Computation and use of torques to compute rotational acceleration
2) Rolling motion and coupled particle-rigid body motion
3) Angular momentum-energy changes for angular momentum conservation
4) Don't need to know section 10.6 (gyroscopes)

Chapter 11

1) Solving translational and rotational equilibrium situations, with friction
2) Center-of-gravity calculations and movement/non-movements of cg
3) Use of stress and strain for linear and volume objects, and liquids
4) Don't need to know section 11.5 (elasticity and plasticity)

Chapter 12

1) Use of Newton's Law of Gravity and Kepler's Three Laws
2) Calculations of kinetic and potential energies with Universal gravity
3) Satellite motion and motion of the planets
4) Definition of black-hole and Schwarzchild radius
5) Don't need to know *proof* of spherical mass result for gravity
6) Don't need to know section 12.7 (effect of Earth's rotation)

REVIEW (Ch. 13): The Pendulum as Simple Harmonic Motion

Exact Differential Equation of Motion for a Pendulum

We have already seen the pendulum motion just in terms of the interchange of kinetic and potential energies in the previous chapters. Now we study the motion of the pendulum from the kinematics point of few. We want to find expression for the position and the velocity as functions of time.

Consider the mass m at the end of the string L to be displaced by the angle θ. There will be a net force $F_{tangential}$ on the mass perpendicular to the direction of the string. The force is just $mg \sin\theta$ (exactly like the weight force down the inclined plane). So we can write

$$F_{tangential} = -mg\sin\theta = ma_{tangential} = m\frac{d^2s}{dt^2}$$

Here s is the tangential displacement along a circular arc: $s = L\theta$ (where θ is in radians). So now Newton's second law is written as

$$F = m\frac{d^2s}{dt^2} = cl\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Approximate Differential Equation of Motion for a Pendulum

The above differential equation of motion is not quite a *restoring force* equation. However, we restrict ourselves to the cases where the angular displacement θ is small such that $\sin\theta \approx \theta$, and where θ is being expressed in radians (not degrees). In that case we have

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

This looks just like the spring equation, where now $\omega^2 = g/L$ instead of $\omega^2 = k/m$. By analogy then, one can write the period of a pendulum as

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad \text{(independent of the mass !)}$$

(Chapter 13) The Physical Pendulum

Besides the **Simple Pendulum** there is also the not–so–simple pendulum which is normally called the **Physical Pendulum**. This is an extended object (rigid body) instead of a point mass which is rotating because of a torque which occurs when the body is displaced from equilibrium. See Figure 13.23 on page 438.
The center-of-mass of the rigid body is at a distance d from a pivot point. The rigid body has a moment of inertia I about an axis through the pivot point. The line from the pivot point to the center-of-mass is at some angle θ with respect to the vertical direction. The weight vector of the rigid body, $m\vec{g}$ can be resolved into a component $mg\cos\theta$ along the line to the pivot point, and a perpendicular component $mg\sin\theta$. It is this perpendicular component which exerts the torque about the pivot point. The basic torque equation $\tau = I\alpha$ becomes (Fig. 13.23)

$$-(mg\sin\theta)d = I\frac{d^2\theta}{dt^2}$$

Essentially the length d in a physical pendulum replaces the string length l of the simple pendulum.
This torque is restoring (negative sign) because it tends to push the rigid body back to equilibrium, that is to reduce angular displacements. We again make the small angle approximation that $\sin\theta \approx \theta$ (in radians) to get

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta = -\omega^2\theta$$

We recognize this as another simple harmonic motion equation for which

$$\omega = \sqrt{\frac{mgd}{I}}$$

and the period is given as usual by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

The textbook has an interesting example of leg of a *Tyrannosaurus Rex* as a physical pendulum on pages 439–440. You can get an approximate value for the walking speed of a *T. Rex* as about 4 miles/hour, the same as a human being. Whether *T. Rex* could run as fast as a jeep, as in the movie *Jurassic Park*, is not discussed.

Chap. 14: Fluid Mechanics (Density, Pressure, and Pascal's Law)

Our study of Mechanics is almost complete. There remains just the subject of *Fluid Mechanics* which applies specifically to gases and liquids for which both the concepts of *pressure* and *density* make sense.
We have already seen in Chapter 1 that the density of an object (solid, liquid, or gas) is the ratio of the mass of the object divided by the volume the object occupies

$$\rho = \frac{m}{V} \tag{14.1}$$

For a tabulation of densities, see Table 14.1 on page 457.

The second quantity of importance in fluid mechanics is *pressure*. Pressure is the force per unit area that a gas or a liquid exerts on the surface of its confining volume. Very simply, pressure is the normal weight or perpendicular per unit area of a fluid or a gas

$$P \equiv \frac{F \perp}{A} \tag{14.3}$$

Pressure is measured in units of **Pascals** (Pa) where 1 Pa is 1 N/m^2. The most important feature about *Pressure* in a liquid or a gas is that the amount of the pressure varies with the depth of the liquid or the gas. The pressure difference between two different levels y_1 and y_2 of a liquid is given by

$$p_2 - p_1 = -\rho g(y_2 - y_1) \tag{14.5}$$

$$p(h) = p_{air} + \rho g h \quad \text{(Pressure at the bottom of a liquid open to air)} \tag{14.8}$$

An important consequence of these equations is that *the pressure is the same for all points of equal depth in any liquid.*

Pascal's Law states that a change in pressure on a fluid is transmitted equally to all points in the fluid and the enclosing surface of the container. This is the principle of the *hydraulic press.*

Variation of Pressure With Depth

Anyone who has dived into a swimming pool, or the ocean, knows that the pressure of the water increases with the depth below the surface. This is true of all liquids, and also of the atmosphere itself. The accumulated weight of the water (or the air) above a person is what is responsible for the pressure exerted at a certain depth. This can be deduced from just the **translational equilibrium** in the y direction. Consider a small, cubical element of fluid in a container, with a cross sectional area A, and a differential height dy. That cubical element has a volume $dV = Ady$, and contains a differential element of mass dm. That volume element is not moving, so:

$$\sum F_y = 0 = \text{Force acting up} - \text{Force acting down}$$

$$\sum F_y = 0 = (P + dP)A - dW - PA = AdP - g(dm) = AdP - g\rho Ady$$

$$\Longrightarrow \frac{dP}{dy} = +\rho g$$

The convention here is that the depth y is measured from the surface of the water, and y increases as you go deeper into the water. So the above equation says that the pressure increases with increasing depth. One can integrate the above derivative equation very simply to get

$$P(y) = \rho g y + \text{ constant}$$

This equations shows clearly that pressure is the same when y is the same. The constant of integration can be determined by knowing the pressure at $y = 0$. If the container is open to the atmosphere, then that pressure $P(y = 0) = P_a$ where P_a is the atmospheric pressure. This gives

$$P(y) = \rho g y + P_a \quad (P_a = 1.01 \text{ x } 10^5 \text{ Pa or } 14.7 \text{ lb/in}^2) \tag{14.8}$$

The pressure at at depth y is equal to the atmospheric pressure plus an amount $\rho g y$ (the weight of a column of unit area with height y).

The Variation of Pressure with Depth

Example of a hydraulic lift

A hydraulic lift consists of a small diameter piston of radius 5 cm, and a large diameter piston of radius 15 cm. How much force must be exerted on the small diameter piston in order to support the weight of a car at 13,300 N ?

The pressure (F/A) on both sides of the hydraulic lift must be the same at the same height y. This lead to

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} \Longrightarrow F_1 = F_2\Big(\frac{A_1}{A_2}\Big)$$

$$F_1 = 13,300\Big(\frac{\pi(0.05)^2}{\pi(0.15)^2}\Big) = 1.48 \text{ x } 10^3 \text{ N}$$

There is a factor of 9 gain in lifting power by means of the hydraulic press. The same force multiplication occurs in the braking system of cars which use brake fluid to transmit the force from the brake pedal.

Second Example

Calculate the pressure at an ocean depth of 1000 m, using the density of water as 1.0 x 10^3 kg/m^3.

From Eq. 14.8 we have

$$P(y = 1000) = \rho g(1000) + 1.01 \text{ x } 10^5 = 9.90 \text{ x } 10^6 \text{ Pa}$$

This pressure is 100 times that of normal atmospheric pressure. Now you know why submarines don't have portholes.

Buoyant Forces and Archimedes' Principle

To continue the swimming pool line of reasoning, many people are able to float in water. This is an example of **buoyancy**, the fact that objects immersed in water weigh less (or nothing) compared to what they weigh out of water. **Archimedes' principle states:**

Any object completely or partially immersed in a fluid is buoyed up by a force equal to the weight of the fluid displaced by the volume occupied by the object.

Worked example

A piece of aluminum ($\rho = 2.7$ x 10^3 kg/m^3) with a mass of 1.0 kg is completely submerged in a container of water. What is the apparent weight of this piece of aluminum ?

The normal weight of the aluminum would be $W = mg = 1.0 \cdot 9.8 = 9.8$ N. When immersed in water, part of that weight is counteracted by the upward buoyant force of the water, B:

$$B = \rho_{water} \cdot g \cdot V_{aluminum} = \rho_{water} \cdot g \cdot \left(\frac{m_{aluminum}}{\rho_{aluminum}}\right) = 1 \text{ x } 10^3 \cdot 9.8 \cdot \left(\frac{1.0}{2.7}\right)$$

$$B = 3.63 \Longrightarrow T_{\text{apparent weight}} = W - B = 6.17 \text{ Newtons}$$

Fluid Dynamics: Equation of Continuity and Bernoulli's Equation

Equations in Fluid Dynmaics
For moving *incompressible fluids* there are two important laws of fluid dynamics:

1) The Equation of Continuity, and
2) Bernoulli's Equation.

These you have to know, and know how to use to solve problems.

The Equation of Continuity
The continuity equation derives directly from the incompressible nature of the fluid. Suppose you have a pipe filled with a moving fluid. If you want to compute the amount of *mass* moving by a point in the pipe, all you need to know is the density ρ of the fluid, the cross sectional area A of the pipe, and the velocity v of the fluid. Then the mass flow is given by $\rho \cdot A \cdot v$ because

$$\rho \cdot A \cdot v = \rho \cdot A \cdot \frac{\Delta x}{\Delta t} = \frac{\rho \Delta V}{\Delta t} = \frac{\Delta m}{\Delta t} \quad \text{(the "mass flow")}$$

If the fluid is truly incompressible, then the mass flow is the same at all points in the pipe, and the density is the same at all points in the pipe:

$$\rho A_1 v_1 = \rho A_2 v_2 \Longrightarrow A_1 v_1 = A_2 v_2 \quad \text{(the equation of continuity)} \qquad (14.10)$$

Bernoulli's Equation
Bernoulli's equation is very powerful equation for moving, incompressible fluids, and can be derived using the conservation of Mechanical Energy. The Bernoulli's Equation states that if you have a fluid moving in a pipe at point 1 with pressure P_1, speed v_1, and height y_1, and the fluid moves to point 2 with pressure P_2, speed v_2, and height y_2, then these six quantities are related as follows

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 \quad \text{(Energy Conservation)}$$

$$P + \frac{1}{2}\rho v^2 + \rho g y = \textbf{ constant} \qquad \text{(Alternate form)} \qquad (14.17)$$

Using Bernoulli's Equation: Venturi Tube and Torricelli's Law

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Worked Example: The Venturi Tube

A horizontal pipe with a constriction is called a *Venturi Tube* and is used to measure flow velocities by measuring the pressure at two different cross sectional areas of the pipe. Given two pressures P_1 and P_2 where the areas are A_2 and A_1 respectively, determine the flow velocity at point 2 in terms of these quantities and the fluid density ρ.

First use Bernoulli's law, and take the heights $y_1 = y_2 = 0$:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Now substitute for one of the velocities, v_1, by using the continuity equation:

$$A_1 v_1 = A_2 v_2 \Longrightarrow v_1 = \frac{A_2}{A_1} v_2$$

$$\Longrightarrow P_1 + \frac{1}{2}\rho\left(\frac{A_2}{A_1} v_2\right)^2 = P_2 + \frac{1}{2}\rho v_2^2 \qquad \Longrightarrow v_2 = A_1\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$$

Example Torricelli's Law (speed of efflux)

A tank with a surface pressure P (at point 2) and a surface area A_2 has a small hole of area $A_1 << A_2$ at a distance of h below the surface. What is the velocity of the escaping fluid which has density ρ ?

$$P_2 + \frac{1}{2}\rho v_2^2 + \rho g h = P_1 + \frac{1}{2}\rho v_1^2 \qquad \text{and} \qquad v_2 = \frac{A_1}{A_2} v_1 \Longrightarrow v_2 \approx 0$$

$$P + \rho g h = P_a + \frac{1}{2}\rho v_1^2 \Longrightarrow v_1 = \sqrt{\frac{2(P - P_a)}{\rho} + 2gh}$$

$$v_1 = \sqrt{2gh} \quad (\text{if } P = P_a)$$

Using Bernoulli's Law

A *large* storage tank filled with water develops a *small* hole in its side at a point 16 m below the water level. If the rate of flow from the leak is 2.5 x 10^{-3} m^3/min, determine

a) the speed at which the water leaves the hole, and

b) the diameter of the hole

Solution

We assume that the tank and the hole are both open to the atmosphere. Call the top position 1 and the point of the hole position 2. So $P_1 = P_2 = P_a$. We now write Bernoulli's law:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

The *continuity equation* allows us to relate the speeds to the areas at the two positions

$$v_1 A_1 = v_2 A_2 \Longrightarrow v_1 = \frac{A_2}{A_1} v_2$$

Because the area $A_1 \gg A_2$ we can ignore v_1 in comparison with v_2 ($v_1 \ll v_2$) Now substitute $v_1 = 0$ and cancel out the equal pressures in Bernoulli's law to get

$$\rho g y_1 = \frac{1}{2}\rho v_2^2 + \rho g y_2 \Longrightarrow v_2^2 = 2g(y_1 - y_2)$$

$$v_2 = \sqrt{2gh} = \sqrt{2 \cdot 9.8 \cdot 16} = 17.7 \text{ m/s}$$

For part b) we know that the volume flow rate is the product of the area of the hole and the velocity

$$\text{flow rate} = Av$$

We first convert the flow rate given in m^3/minute into m^3/second by dividing by 60. This gives 4.167 x 10^{-5} m^3/second

$$4.167 \text{ x } 10^{-5} = A_2 v_2 = A_2 \cdot 17.7 \Longrightarrow A_2 = .2354 \text{ x } 10^{-6} \text{ m}^2$$

This is equivalent to a diameter of 0.0017 meters.