REVIEW: Equation of Continuity and Bernoulli's Equation

Equations in Fluid Dynamics

For moving *incompressible fluids* there are two important laws of fluid dynamics:

1) The Equation of Continuity, and
2) Bernoulli's Equation.

These you have to know, and know how to use to solve problems.

The Equation of Continuity

The continuity equation derives directly from the incompressible nature of the fluid. Suppose you have a pipe filled with a moving fluid. If you want to compute the amount of *mass* moving by a point in the pipe, all you need to know is the density ρ of the fluid, the cross sectional area A of the pipe, and the velocity v of the fluid. Then the mass flow is given by $\rho \cdot A \cdot v$ because

$$\rho \cdot A \cdot v = \rho \cdot A \cdot \frac{\Delta x}{\Delta t} = \frac{\rho \Delta V}{\Delta t} = \frac{\Delta m}{\Delta t} \quad \text{(the “mass flow”)}$$

If the fluid is truly incompressible, then the mass flow is the same at all points in the pipe, and the density is the same at all points in the pipe:

$$\rho A_1 v_1 = \rho A_2 v_2 \Longrightarrow A_1 v_1 = A_2 v_2 \quad \text{(the equation of continuity)} \qquad (14.10)$$

Bernoulli's Equation

Bernoulli's equation is very powerful equation for moving, incompressible fluids, and can be derived using the conservation of Mechanical Energy. The Bernoulli's Equation states that if you have a fluid moving in a pipe at point 1 with pressure P_1, speed v_1, and height y_1, and the fluid moves to point 2 with pressure P_2, speed v_2, and height y_2, then these six quantities are related as follows

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 \quad \text{(Energy Conservation)}$$

$$P + \frac{1}{2}\rho v^2 + \rho g y = \textbf{ constant} \qquad \text{(Alternate form)} \qquad (14.17)$$

Using Bernoulli's Equation: Venturi Tube and Torricelli's Law

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Worked Example: The Venturi Tube
A horizontal pipe with a constriction is called a *Venturi Tube* and is used to measure flow velocities by measuring the pressure at two different cross sectional areas of the pipe. Given two pressures P_1 and P_2 where the areas are A_2 and A_1 respectively, determine the flow velocity at point 2 in terms of these quantities and the fluid density ρ.
First use Bernoulli's law, and take the heights $y_1 = y_2 = 0$:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Now substitute for one of the velocities, v_1, by using the continuity equation:

$$A_1 v_1 = A_2 v_2 \Longrightarrow v_1 = \frac{A_2}{A_1}\, v_2$$

$$\Longrightarrow P_1 + \frac{1}{2}\rho\left(\frac{A_2}{A_1}\, v_2\right)^2 = P_2 + \frac{1}{2}\rho v_2^2 \qquad \Longrightarrow v_2 = A_1\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$$

Example Torricelli's Law (speed of efflux)
A tank with a surface pressure P (at point 2) and a surface area A_2 has a small hole of area $A_1 << A_2$ at a distance of h below the surface. What is the velocity of the escaping fluid which has density ρ ?

$$P_2 + \frac{1}{2}\rho v_2^2 + \rho g h = P_1 + \frac{1}{2}\rho v_1^2 \qquad \text{and} \qquad v_2 = \frac{A_1}{A_2} v_1 \Longrightarrow v_2 \approx 0$$

$$P + \rho g h = P_a + \frac{1}{2}\rho v_1^2 \Longrightarrow v_1 = \sqrt{\frac{2(P - P_a)}{\rho} + 2gh}$$

$$v_1 = \sqrt{2gh} \quad (\text{if } P = P_a)$$

Using Bernoulli's Law

A *large* storage tank filled with water develops a *small* hole in its side at a point 16 m below the water level. If the rate of flow from the leak is 2.5 x 10^{-3} m^3/min, determine

a) the speed at which the water leaves the hole, and

b) the diameter of the hole

Solution

We assume that the tank and the hole are both open to the atmosphere. Call the top position 1 and the point of the hole position 2. So $P_1 = P_2 = P_a$. We now write Bernoulli's law:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

The *continuity equation* allows us to relate the speeds to the areas at the two positions

$$v_1 A_1 = v_2 A_2 \Longrightarrow v_1 = \frac{A_2}{A_1} v_2$$

Because the area $A_1 \gg A_2$ we can ignore v_1 in comparison with v_2 ($v_1 \ll v_2$) Now substitute $v_1 = 0$ and cancel out the equal pressures in Bernoulli's law to get

$$\rho g y_1 = \frac{1}{2}\rho v_2^2 + \rho g y_2 \Longrightarrow v_2^2 = 2g(y_1 - y_2)$$

$$v_2 = \sqrt{2gh} = \sqrt{2 \cdot 9.8 \cdot 16} = 17.7 \text{ m/s}$$

For part b) we know that the volume flow rate is the product of the area of the hole and the velocity

$$\text{flow rate} = Av$$

We first convert the flow rate given in m^3/minute into m^3/second by dividing by 60. This gives 4.167 x 10^{-5} m^3/second

$$4.167 \text{ x } 10^{-5} = A_2 v_2 = A_2 \cdot 17.7 \Longrightarrow A_2 = .2354 \text{ x } 10^{-6} \text{ m}^2$$

This is equivalent to a diameter of 0.0017 meters.

CHAPTER 15: Wave Motion

Qualitative Description of Waves

A **wave** is the propagation of energy (motion) through a medium. When a wave propagates, the medium is disturbed from its equilibrium position for a short period of time and then returns to its normal position. Think of the "wave" which travels around the fans in a football stadium. The medium here is the football fans, and the motion is a pulse movement standing up and then sitting down.

A **transverse wave** propagates in a direction perpendicular to the motion of the medium (again think of the football wave). Water waves are a good example of transverse waves. A **longitudinal wave** has its motion in the direction of the displacement. Sound waves are a good example of longitudinal wave motion.

No matter what the wave, *there is no net displacement of the particles in the medium once the wave has passed.*

Mathematical Description of Waves

A wave is characterized by a displacement y which occurs at a given position x and at a given time t. In order to describe mathematically the equation of a wave, one must write y as a function of *two independent parameters* x and t. The most common type of wave is expressed with a trigonometric sine function.

$$y(x,t) = A\cos(kx - \omega t) \quad \text{(wave ampl. A traveling in the } +x \text{ direction)} \quad (15.3)$$

$$y(x,t) = A\cos(kx + \omega t) \quad \text{(wave ampl. A traveling in the } -x \text{ direction)} \quad (15.4)$$

The most important equation of all is the speed equation for a wave in terms of its frequency f and its wavelength λ

$$v = f\lambda \qquad 15.1$$

The other parameters k and ω are related to λ and f as

$$\text{wave number: } k \equiv \frac{2\pi}{\lambda} \quad \text{and angular frequency: } \omega \equiv 2\pi f$$

For the case of a stretched (guitar) string or cable under a tension F and having a mass M and length L, the speed of a wave pulse is

$$v = \sqrt{\frac{F}{\mu}} \text{ where } \mu \equiv \frac{M}{L} \quad ; \quad v = f\lambda = \frac{\lambda}{T} \text{ (harmonic waves)} \qquad (15.13, 15.1)$$

The Two Types of Waves

Transverse Waves

The most familiar type of wave motion is that of waves on a beach. This motion should give you a good idea of the wave phenomenon. It is the transmission of energy, manifested by the up and down motion of the water, through a medium. Think of a seagull or a duck floating on the water. Before a wave hits, the bird is motionless. Then the bird is successively raised up and lowered by the moving water, and finally the bird goes back to its original height. The same is true of the water itself. Except when the wave is passing through, the molecules of the water are undisturbed from their positions. They occupy the same positions after the wave as before.

This type of water wave is a **transverse wave**. The energy contained in the wave pulse causes the medium to move up and down which is perpendicular to the direction in which the wave pulse is propagating.

Wave motion is an important subject of study because it is the basis for all our electronic communication. Light itself is a wave phenomenon, and the heat from the sun reaches us by *infra–red rays*, which is all part of the same theory of electric and magnetic waves.

Longitudinal Waves

The second type of wave motion is **longitudinal waves**. In this type of wave propagation, the energy of the wave causes the medium to move back and forth in a direction parallel to the direction of propagation of the wave. Sound waves are the most important examples of **longitudinal wave** motion. Another example would be the effect of a gust of wind blowing through a field of wheat. One can see the stalks of wheat “rippling” in the field, moving back and forth, as the gust moves through. Thus the song phrase “amber waves of grain”.

The Equation of Motion of a Traveling Wave in One Dimension

Fundamentally a wave travels. We consider first the simplest case of a *sinusoidal* wave traveling in only one dimension, say x. The displacement that the wave causes we say is in the y direction. The wave is characterized by an amplitude y_m, a wavelength λ and a frequency f, as we saw in the oscillation lecture.

$$\omega = 2\pi f \quad ; \quad T = \frac{1}{f}$$

The displacement y must be a function of both the position x and the time t. If the shape of the wave does not change as it moves along, then we can write a special form of this dependence for a *sinusoidal wave*:

$$y(x, t) = A\cos\left(kx - \omega t\right) \tag{15.7}$$

Fig. 15.4 gives "snapshots" of a traveling wave on the x and the t axes. The parameter k is related to the wavelength $k \equiv 2\pi/\lambda$.

The speed of a traveling wave

The speed of a traveling wave is given by the important formula:

$$v = \lambda f \tag{15.1}$$

This can be obtained by looking at consecutive snapshots of the traveling wave.

The Wave Equation

Previously, for oscillations of a spring or a pendulum, we have written the restoring force equation as a second derivative equation

$$\frac{d^2}{dt^2} = \omega^2 x \Longrightarrow \text{ with solution } \; x(t) = x_m \cos(\omega t + \phi)$$

Wave motion involves *two independent* parameters x and t to produce a dependent function $y(x, t)$. The differential equation of the motion is more complicated:

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial x^2} \Longrightarrow \text{ with solution } \; y(x, t) = A\cos(kx \mp \omega t) \qquad 15.12$$

We can prove the solution works by simply carrying out the derivatives.

Using the Wave Equation

Consider the sinusoidal wave given by the formula

$$y(x,t) = 0.00327 \cos(72.1x - 2.72t)$$

where the three numerical constants are in meters, rad/meter, and rad/s. What is the *amplitude*, the *wavelength*, the *frequency*, and the *speed* of this wave?
To solve this, all one has to do is compare with the basic sinusoidal wave equation:

$$y(x,t) = A \cos(kx - \omega t) \tag{15.7}$$

Then it is simply a matter of comparing the components of this equation with those in the example:

$$\begin{aligned}
\textbf{coefficient of cosine function} &= A = 0.00327 \text{ meters} \\
\textbf{coefficient of x} &= k = 72.1 \text{ rad/m} \quad \Longrightarrow \lambda = \frac{2\pi}{k} = 0.0871 \text{ meters} \\
\textbf{coefficient of t} &= \omega = 2.72 \text{ rad/s} \quad \Longrightarrow f = \frac{\omega}{2\pi} = 0.433 \text{ Hz}
\end{aligned}$$

The wave speed can be computed from $v = \lambda f$

$$v = \lambda f = 0.0871 \cdot 0.433 = 0.0377 \text{ m/s}$$

Lastly, in which direction is this wave traveling: to the right (positive x direction), or to the left (negative x direction)? What one feature to you look at in the expression for this wave?

The Velocity of Waves on a String Under Tension

This being "Music City" we all know about guitars and guitar strings. Guitar strings, which can be metallic or non–metallic, are of varying lengths according to the pitch (frequency) of the sound. The guitar string is tuned to the correct sound by changing slightly the tension using a turn screw in the neck of the guitar at the end of the string. The reason that this works is that the tension of the string determines the velocity of the wave in which is generated when the guitar string is plucked.

$$v = \sqrt{\frac{F}{\mu}} \tag{15.19}$$

where μ is the mass per unit length of the string $\mu = M/L$. By making the tension greater, then the velocity increases. In turn, an increase in velocity leads to a higher frequency f of the sound.

$$v = f\lambda \quad \Longrightarrow \quad f = \frac{v}{\lambda} \tag{15.1}$$

where the **wavelength** of the wave motion in the guitar string is fixed by the length of the string, which we will discuss later.

Worked Example of a Wave Traveling on a String

A uniform string has a mass of 0.3 kg and a length of 6 m. Tension is maintained on the string by suspending a 2 kg mass from one end. Find the speed of a sinusoidal wave in the string:

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{F}{M/L}} = \sqrt{\frac{(2.0 \cdot 9.8)}{0.3/6}} = 19.8 \text{ m/s}$$

Reflection and Transmission of Waves

Another interesting phenomenon about wave motion is what happens when a wave pulse traveling along a string hits a solid wall to which the string is firmly attached. What happens is that the wave is *reflected and inverted*. On the other hand, if a wave is traveling along a string and gets to the end of the string which is not held tight but is free to move, then the wave is *reflected but not inverted.*
A third possibility is the intermediate case. Suppose that there are two strings of different densities which are tied together. A wave pulse is traveling along the first (lighter) string. In that case there will be an inverted reflected inverted wave along the first string, and a non–inverted transmitted wave along the second string. On the other hand, if the wave pulse is first traveling along the heavier string, both the transmitted and the reflected waves will be non–inverted.
When a wave pulse travels from medium A to medium B, and medium B is denser than medium A ($\Longrightarrow v_A > v_B$), then the reflected wave is inverted. Conversely, if medium A is more dense than medium B ($\Longrightarrow v_B > v_A$), then the reflected wave is non–inverted. In either case, the transmitted wave is non–inverted. What about the amplitudes of the reflected and transmitted waves?

Superposition and Interference of Waves

A very interesting facet of wave theory is the **superposition principle**
If two or more traveling waves are moving through a medium, the resultant wave function at any point is the algebraic sum of the wave functions of the individual waves. What this means is that two waves traveling in a string act independently of one another. Two waves can even pass through one another without disturbing their individual shapes. The addition (algebraic sum) of the two waves is called **interference**. For example, if the peak of one wave meets the minimum of a second wave of equal amplitude, then there will be no net displacement of the medium. This is called **destructive interference**. On the other hand if the peak of one wave meets the peak of a second wave, or if the minima of the two waves coincide, then the two waves reinforce each other and the net displacement is doubled. This is called **constructive interference**. When two waves traveling in the same direction, with the same amplitude y_m, the same angular frequency ω, and the same wavelength $\lambda = 2\pi/k$, but separated by a phase difference ϕ meet at the same place, they will add algebraically (superposition)

Superposition and Standing Waves

When two waves traveling in the same direction, with the same amplitude A_0, the same angular frequency ω, and the same wavelength $\lambda = 2\pi/k$, but separated by a phase difference ϕ meet at the same place, they will add algebraically (superposition)

$$y_1(x,t) = A_0 \sin{(kx - \omega t)} \text{ and } y_2(x,t) = A_0 \sin{(kx - \omega t - \phi)}$$

$$y_3 \equiv y_1 + y_2 = A_0 \sin{(kx - \omega t)} + A_0 \sin{(kx - \omega t - \phi)}$$

$$y_3(x,t) = \Big(2A_0 \cos\frac{\phi}{2}\Big) \sin{(kx - \omega t - \frac{\phi}{2})}$$

The phase difference ϕ depends on the location x relative to the source of the two waves. For certain values of x it is possible that $\phi = 0$ in which case the resultant wave has twice the amplitude of the original wave (constructive interference). In other cases, the value of ϕ could be $\pi/2$ or an odd half integer multiple of $\pi/2$ in which case the resultant wave will have zero amplitude (destructive interference). For any value of ϕ in this example, the resultant wave has the same frequency and the same wavelength as the original two waves.

Another example of superposition is to have a string fixed at both ends such that there are waves traveling in opposite directions along the string from reflections at either end. In this case one would have two waves of the form

$$y_1(x,t) = A \cos{(kx - \omega t)} \text{ and } y_2(x,t) = -A \cos{(kx + \omega t)}$$

$$y_3 = y_1 + y_2 = A\left(\cos{(kx - \omega t)} - \cos{(kx + \omega t)}\right) = \Big(2A \sin kx\Big) \sin \omega t$$

This is called a **standing wave** which looks like just a sine function of time but with an amplitude according to the position x. In fact, at certain positions called **nodes**, the amplitude will always be zero (no motion). The node positions are given by $kx = n\pi$ where n is any integer

$$kx = n\pi \Longrightarrow x = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda} = \frac{n\lambda}{2}$$

Since the end of a fixed string at $x = L$ must also be a node, this sets a condition on the wavelengths and frequencies of standing waves therein

$$f_n = \frac{v}{\lambda_n} = \frac{v}{2L/n} = \frac{\sqrt{F/\mu}}{2L/n} = \frac{n}{2L}\sqrt{\frac{F}{\mu}}$$

Standing Wave Problems

Two waves in a long string are given by

$$y_1(x,t) = 0.015\cos{(\frac{x}{2} - 40t)} \quad \text{and} \quad y_2(x,t) = -0.015\cos{(\frac{x}{2} + 40t)}$$

where x, y_1, and y_2 are in meters, and t is in seconds. Determine the positions of the **nodes** of the resulting standing wave, and what is the maximum displacement of the standing wave at the position $x = 0.4$ meters ?

Again you must recognize the components of each individual wave in comparison to the general form: You should see that $A = 0.015$ m, $k = 0.5$ m^{-1}, and $\omega = 40$ s^{-1}, and that y_1 travels to the $+x$ direction, and that y_2 travels in the $-x$ direction.

The superposition of two waves of the same amplitude, wavelength, and frequency, but traveling in opposite directions is a special case, leading to a **standing wave**:

$$\text{standing wave} \quad y_3(x,t) = \Big(2A\sin kx\Big)\sin\omega t$$

$$\text{standing wave} \quad y_3(x,t) = \Big(2\cdot 0.015\sin\frac{x}{2}\Big)\cos 40t = \Big(0.03\sin\frac{x}{2}\Big)\sin 40t$$

The **nodes** of a standing wave are the positions x such that the value of $y_3(x,t)$ is *always* zero, no matter what the value of t. This can only occur when the term in the parenthesis is zero, or the argument of $\sin kx$ is zero. In turn that means that $kx = n\pi$ where n is any integer. In this problem then

$$kx = \frac{x}{2} = n\pi \Longrightarrow x = 2n\pi$$

If one takes a given position $x = 0.4$ m, then the maximum value of $y_3(x = 0.4,\ t)$ will occur when the $\sin\omega t$ function achieves its maximum value which is 1 as usual, specifically when $t = (2n+1)\pi/2\omega$, n=0, 1, ...). In that case

$$y_3(x = 0.4,\ t = \frac{(2n+1)\pi}{2\omega}) = \Big(0.03\sin\frac{0.4}{2}\Big)\sin{(\frac{(2n+1)\pi}{2})} = 0.0294 \text{ m}$$

CHAPTER 16: Sound Waves and Standing Waves

Sound waves are **longitudinal waves** which propagate in a medium such as the air, or perhaps liquids (sonar submarine detection), or even solids. Qualitatively, sound waves are movements of high density (or high pressure) pulses of the medium followed by low density (low pressure) pulses. The high density regions are the regions of **compression**, and the low density regions are the regions of **rarefaction**.

The density pulses occur because parts of the medium shift momentarily from the equilibrium positions giving build–ups and decreases in the normal density. The equation for these position shifts $s(x,t)$:

$$y(x,t) = A\cos(kx - \omega t) \quad \text{where} \quad k \equiv \frac{2\pi}{\lambda} \ \text{ and } \ \omega \equiv 2\pi f = \frac{2\pi}{T} \tag{16.1}$$

Instead of using the displacement $y(x,t)$ to characterize the sound wave, we can also use the pressure change function $p(x,t)$ to characterize the sound wave.

$$p(x,t) = BkA\cos(kx - \omega t) \quad \text{where} \quad \text{B = the Bulk Modulus} \tag{16.4}$$

Here the amplitude of the pressure change wave BkA is computed in terms of the amplitude of the displacement wave y.

The **Doppler Effect** is a well known phenomenon. When a source of sound waves S, say an ambulance with a siren, is approaching ($-v_S$) one hears a higher than normal frequency, and when the ambulance has passed ($+v_S$) the frequency becomes lower. The same effects happen if an observer in a moving car is approaching ($+v_L$ where L means *Listener of sound*) a stationary sound source, and then recedes ($-v_L$) from that sound source. The *frequency heard by the listener* f_L is given in terms of the actual frequency of the source f_S by the following equation

$$f_L = \left(\frac{v \pm v_L}{v \mp v_S}\right) f_S \qquad v = f\lambda \tag{16.26}$$

In the above equation, when the listener is going towards the source you use the $+v_L$ in the numerator. When the listener is going away from the source you use the $-v_L$ in the numerator.
When the source is going towards the listener you use the $-v_S$ in the denominator. When the source is going away from the listener you use the $+v_S$ in the denominator. We will study examples of the Doppler effect next.

The Doppler Effect

The Doppler Effect is an important physical phenomenon which applies to all wave phenomenon, including sound and electromagnetism (*e.g.* radar). Essentially the Doppler effect means that when there is *relative motion* between a source and an observer, then the heard frequency f_L will be different from the emitted frequency from the source f_S. The relationship depends on the velocity v of the wave in the medium, the velocity v_S of the source in the medium, and the velocity v_L of the listener in the medium. Using this notation we have the equation for the perceived frequency in terms of the source frequency:

$$f_L = \left(\frac{v \pm v_O}{v \mp v_S}\right) f_S \quad \text{where} \quad v = f\lambda$$

One uses the $+v_L$ in the numerator when the listener is moving towards the source and $-v_L$ when the observer is moving away from the source. Conversely, one uses the $-v_S$ when the source is moving towards the listener and $+v_S$ when the source is moving away from the listener.

Doppler Effect Problems

Simple Problem A commuter train approaches a passenger platform at a constant speed $v_S = 40$ m/s. The train horn is sounded at 320 Hz. What is the frequency heard by an observer (stationary) on the platform ? What is the wavelength measured by a person on the platform ?

$$\textbf{Doppler Effect} \quad f_L = \left(\frac{v \pm v_L}{v \mp v_S}\right) f_S \qquad v = f\lambda = f'\lambda' = 343 \text{ m/s}$$

Here $v_L = 0$ and we use the $-v_S$ sign since the source is approaching the observer (that makes the frequency f' bigger as experience should tell you)

$$f_L = \left(\frac{343}{343 - 40}\right) 320 = 362 \text{ Hz}$$

The *observed* wavelength is calculated from the *observed* frequency and the unchanged velocity of sound

$$\lambda_L = \frac{v}{f_L} = \frac{343}{362} = 0.948 \text{ meters}$$

Doppler Effect Problems

Harder Problem

A train moving at a speed v_T of 20 m/s is traveling in the same direction as a car which has a speed v_C of 40 m/s. When the car has overtaken and passed the train, then the car horn sounds at 510 Hz (f_1) and the train whistle at 320 Hz (f_2). What is the frequency of the train whistle as heard by the car's occupant, and the car's horn as heard by the train passengers ?

Solution For the people in the car hearing the train whistle

$$f_{2L} = \left(\frac{v - v_C}{v - v_T}\right) 320 = \left(\frac{343 - 40}{343 - 20}\right) 320 = 267 \text{ Hz}$$

Here the source (train, in the denominator) is moving at v_T and it is moving toward the listener (car, in the numerator), and the listener is moving at v_C away from the source. The passengers in the car hear a lowered frequency of the train's whistle.

For the passengers in the train hearing the car's horn

$$f_{1L} = \left(\frac{v + v_T}{v + v_C}\right) 510 = \left(\frac{343 + 20}{343 + 40}\right) 510 = 483 \text{ Hz}$$

Here the source (car, in the denominator) is moving at v_C away from the listener (train, in the numerator) which is moving at v_T toward the source. The train passengers hear a lowered frequency of the car's horn, but the amount of frequency change is not as much.

Standing Sound Waves and Beats

Just as in a fixed string, there can be standing sound waves set up in a column of air. The only difference is that a string normally has both ends fixed, whereas in a column of air one or both ends can be open (to function as a pressure node or a displacement anti node). The result is that there are two sets of equations for the harmonic frequencies:

$$\text{column open at both ends} \qquad f_n = n\frac{v}{2L} \quad (n = 1, 2, 3, ...) \tag{16.16}$$

$$\text{column closed at one end} \qquad f_n = n\frac{v}{4L} \quad (n = 1, 3, 5, ...) \tag{16.22}$$

Beat Frequencies

So far all the superpositions of waves have used the same frequency for both waves. One can also consider what happens when the two waves have different frequencies, say f_1 and f_2 but the same amplitude A. When two such waves are added together the result is more complicated. At a given position x, the sum of the two waves will have a time varying amplitude given by

$$y_3(t) = 2A\cos 2\pi\Big(\frac{f_1 - f_2}{2}\Big)t \cos 2\pi\Big(\frac{f_1 + f_2}{2}\Big)t$$

The human ear can hear the first term of this time varying amplitude as a pulsing sound. These are called *beats: the periodic variation in intensity at a given point due to the superposition of two waves having slightly different frequencies.* If the two frequencies f_1 and f_2 differ by less than 20, then this so–called beat frequency can be heard. One can try striking two piano keys of slightly different pitch to hear the beats. In fact that is how pianos are tuned.