REVIEW: Superposition and Standing Waves

When two waves traveling in the same direction, with the same amplitude A_0, the same angular frequency ω, and the same wavelength $\lambda = 2\pi/k$, but separated by a phase difference ϕ meet at the same place, they will add algebraically (superposition)

$$y_1(x,t) = A_0 \sin(kx - \omega t) \text{ and } y_2(x,t) = A_0 \sin(kx - \omega t - \phi)$$

$$y_3 \equiv y_1 + y_2 = A_0 \sin(kx - \omega t) + A_0 \sin(kx - \omega t - \phi)$$

$$y_3(x,t) = \Big(2A_0 \cos\frac{\phi}{2}\Big) \sin(kx - \omega t - \frac{\phi}{2})$$

The phase difference ϕ depends on the location x relative to the source of the two waves. For certain values of x it is possible that $\phi = 0$ in which case the resultant wave has twice the amplitude of the original wave (constructive interference). In other cases, the value of ϕ could be $\pi/2$ or an odd half integer multiple of $\pi/2$ in which case the resultant wave will have zero amplitude (destructive interference). For any value of ϕ in this example, the resultant wave has the same frequency and the same wavelength as the original two waves.

Another example of superposition is to have a string fixed at both ends such that there are waves traveling in opposite directions along the string from reflections at either end. In this case one would have two waves of the form

$$y_1(x,t) = A \cos(kx - \omega t) \text{ and } y_2(x,t) = -A \cos(kx + \omega t)$$

$$y_3 = y_1 + y_2 = A\left(\cos(kx - \omega t) - \cos(kx + \omega t)\right) = \Big(2A \sin kx\Big) \sin \omega t$$

This is called a **standing wave** which looks like just a sine function of time but with an amplitude according to the position x. In fact, at certain positions called **nodes**, the amplitude will always be zero (no motion). The node positions are given by $kx = n\pi$ where n is any integer

$$kx = n\pi \Longrightarrow x = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda} = \frac{n\lambda}{2} \quad (n = 1, 2, 3, \ldots)$$

Since the end of a fixed string at $x = L$ must also be a node, this sets a condition on the wavelengths and frequencies of standing waves therein

$$L = \frac{n\lambda}{2} \Longrightarrow f_n = \frac{v}{\lambda_n} = \frac{v}{2L/n} = \frac{\sqrt{F/\mu}}{2L/n} = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \quad (n = 1, 2, 3, \ldots)$$

ActivPhysics OnLine Example 10.4

Initial Simulation Conditions and Questions 1–4

The *ActivPhysics 10.4 Example* simulation introduces standing waves in a 1.0 meter long string under tension. A tension value $T = 1.6$ N is to be selected, along with a mass per unit length paramemter $\mu = 0.1$ kg/m. With these two parameters, the speed of a wave in the string is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{1.6}{0.1}} = 4.0 \text{ m/s}$$

From the previous page, we will now that the frequencies of the standing waves in the string are given by

$$f_n = \frac{v}{2L/n} = n\frac{4.0}{2} = 2n \text{ Hz}$$

The fundmental frequency has $n = 1$, which in this case is $f_1 = 2.0$ Hz. The harmonics occur for $n = 2, 3, \ldots$ meaning $f_2 = 4.0$ Hz, $f_3 = 6.0$ Hz, and so on. We can dial those frequencies in to see the various standing wave patterns.

Change of Parameters for Questions 5–6

The second set of simulations in this example sets the tension $T = 3.0$ N, and the mass per unit length as $\mu = 0.03$ kg/m. With these parameters we get a different speed for the waves in the string

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{3.0}{0.03}} = 10.0 \text{ m/s}$$

In turn, this gives a new set of standing wave frequencies

$$f_n = \frac{v}{2L/n} = n\frac{10.0}{2} = 5n \text{ Hz}$$

We can dial in these numbers to see what happens.

Last Simulation, Question 7

The final simulation in this example has the tension at $T = 3.6$ N, and the mass parameter at $\mu = 0.1$ kg/m. With these parameters we get a third speed

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{3.6}{0.1}} = 6.0 \text{ m/s} \Longrightarrow f_n = n\frac{6.0}{2} = 3n \text{ Hz}$$

Standing Wave Problems

Two waves in a long string are given by

$$y_1(x,t) = 0.015\cos\left(\frac{x}{2} - 40t\right) \quad \text{and} \quad y_2(x,t) = -0.015\cos\left(\frac{x}{2} + 40t\right)$$

where x, y_1, and y_2 are in meters, and t is in seconds. Determine the positions of the **nodes** of the resulting standing wave, and what is the maximum displacement of the standing wave at the position $x = 0.4$ meters ?
Again you must recognize the components of each individual wave in comparison to the general form: You should see that $A = 0.015$ m, $k = 0.5$ m^{-1}, and $\omega = 40$ s^{-1}, and that y_1 travels to the $+x$ direction, and that y_2 travels in the $-x$ direction.
The superposition of two waves of the same amplitude, wavelength, and frequency, but traveling in opposite directions is a special case, leading to a **standing wave**:

$$\text{standing wave} \quad y_3(x,t) = \Big(2A\sin kx\Big)\sin\omega t$$

$$\text{standing wave} \quad y_3(x,t) = \Big(2\cdot 0.015\sin\frac{x}{2}\Big)\cos 40t = \Big(0.03\sin\frac{x}{2}\Big)\sin 40t$$

The **nodes** of a standing wave are the positions x such that the value of $y_3(x,t)$ is *always* zero, no matter what the value of t. This can only occur when the term in the parenthesis is zero, or the argument of $\sin kx$ is zero. In turn that means that $kx = n\pi$ where n is any integer. In this problem then

$$kx = \frac{x}{2} = n\pi \Longrightarrow x = 2n\pi$$

If one takes a given position $x = 0.4$ m, then the maximum value of $y_3(x = 0.4,\ t)$ will occur when the $\sin\omega t$ function achieves its maximum value which is 1 as usual, specifically when $t = (2n+1)\pi/2\omega$, with n=0, 1, ... In that case

$$y_3\left(x = 0.4,\ t = \frac{(2n+1)\pi}{2\omega}\right) = \Big(0.03\sin\frac{0.4}{2}\Big)\sin\left(\frac{(2n+1)\pi}{2}\right) = 0.0294 \text{ m}$$

CHAPTER 16: Sound Waves and Standing Waves

Sound waves are **longitudinal waves** which propagate in a medium such as the air, or perhaps liquids (sonar submarine detection), or even solids. Qualitatively, sound waves are movements of high density (or high pressure) pulses of the medium followed by low density (low pressure) pulses. The high density regions are the regions of **compression**, and the low density regions are the regions of **rarefaction**.

The density pulses occur because parts of the medium shift momentarily from the equilibrium positions giving build–ups and decreases in the normal density. The equation for these position shifts $s(x,t)$:

$$y(x,t) = A\cos{(kx - \omega t)} \quad \text{where} \quad k \equiv \frac{2\pi}{\lambda} \text{ and } \omega \equiv 2\pi f = \frac{2\pi}{T} \tag{16.1}$$

Instead of using the displacement $y(x,t)$ to characterize the sound wave, we can also use the pressure change function $p(x,t)$ to characterize the sound wave.

$$p(x,t) = BkA\cos{(kx - \omega t)} \quad \text{where} \quad \text{B = the Bulk Modulus} \tag{16.4}$$

Here the amplitude of the pressure change wave BkA is computed in terms of the amplitude of the displacement wave y.

The **Doppler Effect** is a well known phenomenon. When a source of sound waves S, say an ambulance with a siren, is approaching ($-v_S$) one hears a higher than normal frequency, and when the ambulance has passed ($+v_S$) the frequency becomes lower. The same effects happen if an observer in a moving car is approaching ($+v_L$ where L means *Listener of sound*) a stationary sound source, and then recedes ($-v_L$) from that sound source. The *frequency heard by the listener* f_L is given in terms of the actual frequency of the source f_S by the following equation

$$f_L = \left(\frac{v \pm v_L}{v \mp v_S}\right) f_S \qquad v = f\lambda \tag{16.26}$$

In the above equation, when the listener is going towards the source you use the $+v_L$ in the numerator. When the listener is going away from the source you use the $-v_L$ in the numerator.

When the source is going towards the listener you use the $-v_S$ in the denominator. When the source is going away from the listener you use the $+v_S$ in the denominator. We will study examples of the Doppler effect next.

The Doppler Effect

The Doppler Effect is an important physical phenomenon which applies to all wave phenomenon, including sound and electromagnetism (*e.g.* radar). Essentially the Doppler effect means that when there is *relative motion* between a source and an observer, then the heard frequency f_L will be different from the emitted frequency from the source f_S. The relationship depends on the velocity v of the wave in the medium, the velocity v_S of the source in the medium, and the velocity v_L of the listener in the medium. Using this notation we have the equation for the perceived frequency in terms of the source frequency:

$$f_L = \left(\frac{v \pm v_L}{v \mp v_S} \right) f_S \quad \text{where} \quad v = f\lambda$$

One uses the $+v_L$ in the numerator when the listener is moving towards the source and $-v_L$ when the observer is moving away from the source. Conversely, one uses the $-v_S$ when the source is moving towards the listener and $+v_S$ when the source is moving away from the listener.

Doppler Effect Problems

Simple Problem A commuter train approaches a passenger platform at a constant speed $v_S = 40$ m/s. The train horn is sounded at 320 Hz. What is the frequency heard by an observer (stationary) on the platform ? What is the wavelength measured by a person on the platform, assuming a speed of sound at 343 m/s?

$$\textbf{Doppler Effect} \quad f_L = \left(\frac{v \pm v_L}{v \mp v_S} \right) f_S \qquad v = f\lambda = f'\lambda' = 343 \text{ m/s}$$

Here $v_L = 0$ and we use the $-v_S$ sign since the source is approaching the observer (that makes the frequency f' bigger as experience should tell you)

$$f_L = \left(\frac{343}{343 - 40} \right) 320 = 362 \text{ Hz}$$

The *observed* wavelength is calculated from the *observed* frequency and the unchanged velocity of sound

$$\lambda_L = \frac{v}{f_L} = \frac{343}{362} = 0.948 \text{ meters}$$

Doppler Effect Problems

Harder Problem

A train moving at a speed $v_T = 20$ m/s is traveling in the same direction as a car which has a speed $v_C = 40$ m/s. When the car has overtaken and passed the train, then the car horn sounds at 510 Hz ($= f_1$) and the train whistle at 320 Hz ($= f_2$). What is the frequency of the train whistle as heard by the car's occupant, and the car's horn as heard by the train passengers, assuming a speed of sound at 343 m/s?

Solution

For the people in the car hearing the train whistle

$$f_{2L} = \left(\frac{v - v_L}{v - v_S}\right) f_{2S} = \left(\frac{v - v_C}{v - v_T}\right) 320 = \left(\frac{343 - 40}{343 - 20}\right) 320 = 300 \text{ Hz}$$

Here the source (train, in the denominator) is moving at v_T and it is moving toward the listener (car, in the numerator), and the listener is moving at v_C away from the source. The passengers in the car hear a lowered frequency of the train's whistle.

For the passengers in the train hearing the car's horn

$$f_{1L} = \left(\frac{v + v_L}{v + v_S}\right) f_{1S} = \left(\frac{v + v_T}{v + v_C}\right) 510 = \left(\frac{343 + 20}{343 + 40}\right) 510 = 483 \text{ Hz}$$

Here the source (car, in the denominator) is moving at v_C away from the listener (train, in the numerator) which is moving at v_T toward the source. The train passengers hear a decreased frequency of the car's horn.

Standing Sound Waves and Beats

Just as in a fixed string, there can be standing sound waves set up in a column of air. The only difference is that a string normally has both ends fixed, whereas in a column of air one or both ends can be open (to function as a pressure node or a displacement anti node). The result is that there are two sets of equations for the harmonic frequencies:

column open at both ends $$f_n = n\frac{v}{2L} \quad (n = 1, 2, 3, ...) \tag{16.16}$$

column closed at one end $$f_n = n\frac{v}{4L} \quad (n = 1, 3, 5, ...) \tag{16.22}$$

Beat Frequencies

So far all the superpositions of waves have used the same frequency for both waves. One can also consider what happens when the two waves have different frequencies, say f_1 and f_2 but the same amplitude A. When two such waves are added together the result is more complicated. At a given position x, the sum of the two waves will have a time varying amplitude given by

$$y_3(t) = 2A\cos 2\pi\Big(\frac{f_1 - f_2}{2}\Big)t \cos 2\pi\Big(\frac{f_1 + f_2}{2}\Big)t$$

The human ear can hear the first term of this time varying amplitude as a pulsing sound. These are called *beats: the periodic variation in intensity at a given point due to the superposition of two waves having slightly different frequencies.* If the two frequencies f_1 and f_2 differ by less than 20, then this so–called beat frequency can be heard. One can try striking two piano keys of slightly different pitch to hear the beats. In fact that is how pianos are tuned.