# **REVIEW: Vectors and Scalars**

Some quantities in physics such as mass, length, or time are called *scalars*. A quantity is a scalar if it obeys the ordinary mathematical rules of addition and subtraction. All that is required to specify these quantities is a magnitude expressed in an appropriate units.

A very important class of physical quantities are specified not only by their magnitudes, but also by their *directions*. Perhaps the most important of these quantities is **FORCE**. Consider a heavy trunk on a smooth (almost slippery) floor, weighing say 100 pounds. You want to move the trunk but you are only able to lift 50 pounds.

# What do you do?

A **vector** must always be specified by giving its magnitude and direction. In turn the vector's direction must be given with respect to some known direction such as the horizontal or the vertical direction, or perhaps with respect to some pre-defined "X" axis.

The specification of the magnitude and direction does not have to be direct or explicit. The specification can be indirect or implicit by giving the "X" and "Y" **components** of the vector, and it is up to you to use the Pythagorean theorem to calculate the actual magnitude and direction.

(Do you remember your trigonometry?)

- 1) What is a right triangle? How many *degrees* are there in a triangle?
- 2) What are the definitions of **sine**, **cosine**, and **tangent**?
- 3) What is the Pythagorean theorem ?
- 4) What is the *law of sines* ?
- 5) What is the *law of cosines*?
- 6) What is a *radian* ?

# **REVIEW:** Multiplication of a Vector by a Scalar

The easiest type of multiplication is that of a vector by a scalar. This always produces another vector.

Be careful – multiplication of a vector by a scalar is **NOT** the same as scalar multiplication of two vectors which is covered in Chapter 6.

There are two possibilities depending on whether the scalar is dimensionless or has dimensions. If the scalar is dimensionless, that is a pure number, than the multiplication simply enlarges (or reduces) the original vector if the scalar is bigger (or less) than 1.0.

If the scalar has dimensions, then the result is a new type of vector because the product will have different dimensions than the original vector. So it does not make any sense to say "larger" or "smaller" in this type of multiplication. One of the most important laws in mechanics involves the multiplication of a vector by a scalar. This is *Newton's Second Law of Motion* given by:

$$\vec{F} = m\vec{a}$$

Here  $\vec{F}$  is a *force* vector and  $\vec{a}$  is an *acceleration* vector. The scalar m is the mass. All three of these physical quantities have different dimensions, and this multiplication of a vector by a scalar effectively turns an acceleration vector into a force vector.

The usefulness of the vector notation comes about when it is realized that a simple equation such as Newton's Second Law is actually three separate equations:

$$\vec{F} = m\vec{a} \Longrightarrow F_x \,\hat{\mathbf{i}} = ma_x \,\hat{\mathbf{i}} \Longrightarrow F_y \,\hat{\mathbf{j}} = ma_y \,\hat{\mathbf{j}} \Longrightarrow F_z \,\hat{\mathbf{k}} = ma_z \,\hat{\mathbf{k}}$$

This means that the motions in the x, y, and z directions are independent of one another, and the study of the problem of motion in one, two, or three dimensions can be very much simplified.

## **REVIEW:** Analytic Addition of Vectors using Vector Components

The graphical addition of vectors is not terribly convenient, especially if a numerical solution is required. Much more often you will have to add vectors *analytically*. By that is meant that you first *resolve* the vectors into their perpendicular components, then add the components by ordinary mathematics, and finally reconstitute the resultant with trigonometry and the Pythagorean theorem.

Instead of one vector  $\vec{\mathbf{A}}$ , now take another vector  $\vec{\mathbf{B}}$ Let's say  $\vec{\mathbf{A}}$  is at an angle  $\alpha$ , and  $\vec{\mathbf{B}}$  is at an angle  $\beta$ The vector sum of  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  is denoted by  $\vec{\mathbf{R}}$ 

$$\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$$

This can be solved component-by-component

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$

Now solve for  $\vec{\mathbf{R}}$ . First the magnitude

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

And now the *direction* of  $\vec{\mathbf{R}}$  which we symbolize as  $\gamma$ 

$$\gamma = \tan^{-1} \frac{R_y}{R_x} = \tan^{-1} \frac{(A_y + B_y)}{(A_x + B_x)}$$

# **REVIEW:** Analytic Addition of Vectors using Vector Components

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# TABLE FORM OF ANALYTIC ADDITION

Vector	Angle	Magn. of $x$ component	Magn. of $y$ component
$\vec{\mathbf{A}}$	α	$A\cos \alpha$	$A\sin \alpha$
B	β	$B\cos\beta$	$B\sin\beta$
$\vec{\mathbf{R}}$	$\gamma = \tan^{-1} \frac{R_y}{R_x}$	$R_x = A\cos\alpha + B\cos\beta$	$R_y = A\sin\alpha + B\sin\beta$

## Worked Example

A hiker walks 25 km due southeast  $(= -45^{\circ})$  the first day, and 40 km at  $60^{\circ}$  north of east. What is her total displacement for the two days?

Arrange the problem in the table above with  $\vec{\mathbf{A}}$  being the first day's displacement and  $\vec{\mathbf{B}}$  being the second day's displacement:

Vector	Angle	Magn. of $x$ component	Magn. of $y$ component
(km)	( <sup>O</sup> )	$(\mathrm{km})$	$(\mathrm{km})$
A = 25	-45	$A\cos\left(-45\right) =$	$A\sin\left(-45\right) =$
B = 40	+60	$B\cos\left(+60\right) =$	$B\sin\left(+60\right) =$
R =	$\gamma = \tan^{-1} \frac{R_y}{R_x} =$	$R_x =$	$R_y =$

# **REVIEW:** Analytic Addition of Vector Components

# Worked Example

You are given a displacement of 20 km to the West, and a second displacement at 10 km to the North. What is the sum of the two displacements?

Vector	Angle	Magn. of $x$ component	Magn. of $y$ component
(km)	( <sup>0</sup> )	$(\mathrm{km})$	$(\mathrm{km})$
Vector	Angle	Magn. of $x$ component	Magn. of $y$ component
(km)	$(^{\mathrm{O}})$	(km)	(km)
A = 20	+180	$A\cos\left(+180\right) =$	$A\sin\left(+180\right) =$
B = 10	+90	$B\cos\left(+90\right) =$	$B\sin\left(+90\right) =$
R =	$\gamma = \tan^{-1} \frac{R_y}{R_x} =$	$R_x =$	$R_y =$

 $\mathbf{WARNING} \ \mathrm{Know} \ \mathrm{your} \ \mathrm{quadrants} \ !!$ 

## **CHAPTER 2:** Motion in One Dimension

**Kinematics** is the study of motion by relating the position of a particle to the instant of time for which the particle is at that position.

The relevant (vector) physical quantities for kinematics are:

1) Displacement ( $\equiv \Delta X = x_2 - x_1$  in one dimension)

2) Velocity (average or instantaneous)

3) Acceleration (average or instantaneous)

The **average velocity** of a particle is the displacement of a particle divided by the time over which that displacement occurred:

$$\vec{\mathbf{v}} \equiv \frac{\Delta \vec{x}}{\Delta t} = \frac{\vec{x_2} - \vec{x_1}}{t_2 - t_1}$$

The average velocity is always defined over some specified time interval.

The **instantaneous velocity** is the limit of the average velocity as the time interval approaches 0.

$$\mathbf{v}(\mathbf{\tilde{t}}) \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{x}}{\Delta t}$$

Note: the instantaneous velocity is always defined at a specific time instant.

Acceleration: One proceeds one step further, and defines the average and instantaneous *accelerations* exactly analogously to the velocity definitions. The **average acceleration** of a particle is the change in the particles velocity divided by the time over which that velocity change occurred

$$\overline{\vec{\mathbf{a}}} \equiv \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v_2} - \vec{v_1}}{t_2 - t_1}$$

The average acceleration is always defined over some specified time interval. The **instantaneous acceleration** is the limit of the average acceleration as the time interval approaches 0.

$$\mathbf{a}(\mathbf{\vec{t}}) \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

Note: the instantaneous acceleration is always defined at a specific time instant.

# Definition of AVERAGE VELOCITY

You leave Vanderbilt one Saturday afternoon in your car on the way to Knoxville to see VU play UT basketball at 8:00 P.M. Your departure time is exactly 5:00 P.M. so you're running a little late (because of all that extra studying.) Unfortunately, at exactly 6:30 P.M. just past Cookeville on I40, you see those blue lights flashing in your rear view mirror. You have been clocked doing 75 miles/hour as a blip on a radar gun. You begin to plan your defense.

## A displaced person

One dimensional motion (I40 is mostly East–West), call the coordinate **X** Initial position, Nashville,  $X_i$ , Final position, Cookeville,  $X_f$ Your displacement vector then is:

$$\Delta \vec{X} = \vec{X_f} - \vec{X_i}$$

What is the direction of your displacement vector ? What is the magnitude of your displacement vector ? Nashville-to-Cookeville = 80 miles (Exit 208 to Exit 288).  $\implies$  Displacement vector = 80 miles, East

Your average velocity  $\vec{V}$ 

$$\vec{\mathbf{V}} \equiv \frac{\Delta \vec{\mathbf{X}}}{\Delta \mathbf{t}} = \frac{\vec{X_f} - \vec{X_i}}{t_f - t_i} = \frac{80 \text{ miles, East}}{1.5 \text{ hours}} = 53.3 \frac{\text{miles}}{\text{hour}}$$
, East

The average velocity

- 1) must always specify a particular time interval
- 2) is the ratio of the displacement vector over the time interval
- 3) has the same *direction* as the displacement vector

#### The defense

So the radar gun must have been wrong because you can prove that your average velocity was only 53.3 miles/hour. Not only are you law-abiding, but you are also energy conscious. (The judge knows kinematics, you lose.)

#### The Instantaneous Velocity

The **average velocity**, or the average speed, does not tell us anything about the speed or velocity at any particular time. It is a physical quantity which is computed over an interval of time. In fact, if you had returned to Nashville for some reason, your average velocity would be **zero**. *Why* ?

In order to know the speed or velocity at any given instant of time, we must have another quantity the **instantaneous velocity** 

The **instantaneous velocity** is the limit of the average velocity as the time interval approaches 0.

$$\mathbf{v}(\mathbf{\vec{t}}) \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{\vec{x}}}{\Delta t}$$

Note: the instantaneous velocity is always defined at a specific time instant. Putting that calculus to work, you recognize what that limit really is

$$\mathbf{v}(\mathbf{\vec{t}}) \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{\vec{x}}}{\Delta t} = \frac{d\mathbf{\vec{x}}}{dt}$$

In one dimension we can drop the vector notation, and rewrite this for the average speed:

$$v(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

The instantaneous speed is defined at a particular instant of time. It is equal to the *slope* of the position function x(t). In order to calculate the instantaneous speed, you need to know the position function, x(t), at each and every instant of time t. This can be best shown by having a graph of x(t) as a function of t.

#### Average and Instantaneous Velocity

Worked Example A particle moves along the x axis. Its x coordinate in meters varies with time t in seconds according to

$$x(t) = -4t + 2t^2$$

a) What is the displacement of the particle over the interval from t = 0 to t = 1? second, and over the interval from t = 1 to t = 3

**Solution** For each time interval we calculate the position vector at the beginning and at the end of the interval, and then we take the difference:

$$\Delta \vec{x}_{01} \equiv \vec{x}_1 - \vec{x}_0$$

$$\vec{x}_0 = \vec{x}(t=0) = (-4(0) + 2(0)^2)\,\hat{\mathbf{i}} = 0$$

and 
$$\vec{x}_1 = \vec{x}(t=1) = (-4(1) + 2(1)^2) \,\hat{\mathbf{i}} = -2 \,\hat{\mathbf{i}}$$

Take the difference:  $\vec{x}_1 = -2\hat{\mathbf{i}}$  less  $\vec{x}_0 = 0 \Longrightarrow \Delta \vec{x}_{01} = -2\hat{\mathbf{i}}$  meters

Next calculate 
$$\Delta \vec{x}_{13} \equiv \vec{x}_3 - \vec{x}_1$$
  
 $\vec{x}_3 = \vec{x}(t=3) = (-4(3) + 2(3)^2) \,\hat{\mathbf{i}} = 6 \,\hat{\mathbf{i}}$   
 $\Longrightarrow \Delta \vec{x}_{13} = 6 \,\hat{\mathbf{i}} - (-2) \,\hat{\mathbf{i}} = 8 \,\hat{\mathbf{i}}$  meters

b) What is the average velocity during the above two time intervals?

**Solution** The average velocity is the average displacement divided by the time over which that displacement occurred:

$$\overline{\vec{v}}_{01} = \frac{\Delta \vec{x}_{01}}{1-0} = \frac{-2\,\hat{\mathbf{i}}}{1} - 2\,\hat{\mathbf{i}} \,\mathrm{m/s}$$

Similarly, for the interval from t = 1 to t = 3 seconds:

$$\overline{\vec{v}}_{13} = \frac{\Delta x_{13}}{3-1} = \frac{+8\,\hat{\mathbf{i}}}{2} + 4\,\hat{\mathbf{i}} \,\mathrm{m/s}$$

What is the instantaneous velocity at t = 2.5 seconds?

**Solution** We must take the tangent of the x versus t plot at the value of t = 2.5 seconds in order to get the instantaneous velocity. Depending on how good we align that slope, we should find that  $\vec{v} = +6\hat{i}$  m/s.



Figure 1: a) The upper graph shows the position of a particle as a function of time where the particle's position is given by the function  $x(t) = -4t + 2t^2$ . b) The lower graph shows the speed of the same particle, where the equation for the speed is obtained as the first time derivative of the position function.

#### Instantaneous Velocity

We have seen that the instantaneous velocity in one dimension is given by

$$v(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Let us now apply this definition to the previous example where we had

$$x(t) = -4t + 2t^2$$

## **Derivative Calculation**

Taking the derivative should be easy enough

$$v(t) \equiv \frac{dx}{dt} = -4 + 4t$$

If we want the speed at t = 2.5 seconds, we just substitute in the formula:

$$\implies v(t = 2.5) = -4 + 4(2.5) = 6 \text{ m/s}$$

## **Explicit Limit Calculation**

For the explicit limit calculation we will start at an initial time t and go to a final time  $t + \Delta t$ . We have to calculate  $\Delta x$  in this interval where we symbolize the initial value of x by  $x_i$  and the final value of x by  $x_f$ . We get first

$$x_i(t) = -4t + 2t^2$$

The final value of  $x_f$  is given by

$$x_f(t + \Delta t) = -4(t + \Delta t) + 2(t + \Delta t)^2 = -4t - 4\Delta t + 2t^2 + 4t\Delta t + 2(\Delta t)^2$$

The displacement  $\Delta x$  is then given by

$$\Delta x = x_f(t + \Delta t) - x_i(t)$$
  
$$\Delta x = -4t - 4\Delta t + 2t^2 + 4t\Delta t + 2(\Delta t)^2 - (4t + 2t^2)$$
  
$$\Delta x = -4\Delta t + 4t\Delta t + 2(\Delta t)^2$$

Now we divide this  $\Delta x$  by  $\Delta t$  to get the average velocity

$$\overline{v} \equiv \frac{\Delta x}{\Delta t} = \frac{-4\Delta t + 4t\Delta t + 2(\Delta t)^2}{\Delta t} = -4 + 4t + 2\Delta t$$

Finally, we take the limit as  $\Delta t \to 0$ 

$$\lim_{\Delta t \to 0} \overline{v} = \lim_{\Delta t \to 0} (-4 + 4t + 2\Delta t) = -4 + 4t$$

This is the same expression as we obtained directly above using calculus.

## Acceleration

We have seen that the **velocity** of a particle is the time rate of change of the position. There is also a physical quantity which is the time rate of change of velocity. That quantity is called the **acceleration** 

# Average acceleration

The average acceleration is analogous to the average velocity:

$$\overline{a_{if}} \equiv \frac{v_f - v_i}{t_f - t_i} = \frac{\Delta v}{\Delta t}$$

Like the average velocity, the average acceleration must be over a specified time interval and not a a particular time instant.

# Instantaneous acceleration

The instantaneous acceleration is analogous to the instantaneous velocity:

$$a(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

By your knowledge of calculus, you can re-write this as

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = \frac{d^2x(t)}{dt^2}$$

In order to compute the instantaneous acceleration, you need to know the velocity function of time v(t), or the position function of time x(t).

# Remember

Velocity is the first time derivative of the position function x(t)

Acceleration is the first time derivative of the velocity function v(t)Acceleration is therefore the second time derivative of the position function x(t)

Almost all the problems you will work out in the beginning of the semester have either zero for the acceleration, or a constant value for the acceleration. In particular you will most often have to solve problems where the constant acceleration is that due to gravity.

# $\label{eq:constant} \begin{array}{c} CONSTANT \ Acceleration \ Equations \ of \ Motion \\ Velocity \ equation \ v(t) \ in \ terms \ of \ v_0 \ and \ a \end{array}$

If the acceleration is constant, then the average value of the acceleration equals the instantaneous value of the acceleration. So, for constant acceleration a over an interval t = 0 to some final value t = t

$$a(t) = a = \frac{v(t) - v(t=0)}{t} \Longrightarrow v(t) = v_0 + at$$
 2.8

The speed at a time t equals the initial speed plus the product of the constant acceleration times the elapsed time.

# Position Equation x(t) in terms of $v_0$ and v(t)

With constant, non-zero acceleration, the velocity changes with time according to the previous equation. However the velocity is linear with time, so that means that the average velocity over a time interval is one-half the initial plus the final velocities:

$$\overline{v_{if}} = \frac{1}{2}(v_f + v_i) \tag{2.10}$$

This is true for any time interval, as long as the acceleration remains constant. Let us now choose a particular time interval,  $t_i = 0$ , and  $t_f = t$ , and re-write the above as:

$$\overline{v} = \frac{1}{2}(v(t) + v_0)$$

But we know that the average velocity is also the rate of change of the position function x(t)

$$\overline{v} = \frac{x(t) - x_0}{t}$$

Equating these two equations one will obtain

$$x(t) - x_0 = \frac{1}{2}(v(t) + v_0)t$$

We then substitute for v from Eq. 2.7 into Eq. 2.9 to obtain

$$x(t) - x_0 = \frac{1}{2}(v_0 + at + v_0)t \Longrightarrow$$
  
$$x(t) - x_0 = v_0t + \frac{1}{2}at^2$$
  
2.12

# The Third Kinematic Equation for Constant Acceleration

What result will you get by eliminating the time t parameter given the two previously derived equations 2.8 and 2.12?