REVIEW: Acceleration

We have seen that the **velocity** of a particle is the time rate of change of the position. There is also a physical quantity which is the time rate of change of velocity. That quantity is called the **acceleration**

Average acceleration

The average acceleration is analogous to the average velocity:

$$\overline{a_{if}} \equiv \frac{v_f - v_i}{t_f - t_i} = \frac{\Delta v}{\Delta t} \qquad 2.4$$

Like the average velocity, the average acceleration must be over a specified time interval and not a a particular time instant.

Instantaneous acceleration

The instantaneous acceleration is analogous to the instantaneous velocity:

$$a(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \qquad 2.5$$

By your knowledge of calculus, you can re–write this as

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}(\frac{dx(t)}{dt}) = \frac{d^2x(t)}{dt^2}$$

In order to compute the instantaneous acceleration, you need to know the velocity function of time $v(t)$, or the position function of time $x(t)$.

Remember

Velocity is the first time derivative of the position function $x(t)$
Acceleration is the first time derivative of the velocity function $v(t)$
Acceleration is therefore the second time derivative of the position function $x(t)$

Almost all the problems you will work out in the beginning of the semester have either zero for the acceleration, or a constant value for the acceleration.
In particular you will most often have to solve problems where the constant acceleration is that due to gravity.

REVIEW: **CONSTANT Acceleration** Equations of Motion

Velocity equation v(t) in terms of $\mathbf{v_0}$ and a

If the acceleration is constant, then the average value of the acceleration equals the instantaneous value of the acceleration. So, for constant acceleration a over an interval $t = 0$ to some final value $t = t$

$$a(t) = a = \frac{v(t) - v(t = 0)}{t} \Longrightarrow v(t) = v_0 + at \qquad 2.8$$

The speed at a time t equals the initial speed plus the product of the constant acceleration times the elapsed time.

Position Equation x(t) in terms of $\mathbf{v_0}$ and v(t)

With constant, non–zero acceleration, the velocity changes with time according to the previous equation. Since the velocity is linear with time, then the average velocity over a time interval is one–half the initial plus final velocities:

$$\overline{v_{if}} = \frac{1}{2}(v_f + v_i) \qquad 2.10$$

This is true for any time interval, as long as the acceleration remains constant. Let us now choose a particular time interval, $t_i = 0$, and $t_f = t$ and re–write:

$$\overline{v} = \frac{1}{2}(v(t) + v_0) \quad \text{and also} \quad \overline{v} = \frac{x(t) - x_0}{t}$$

Putting these two equations equal to each other produces the equation:

$$x(t) - x_0 = \frac{1}{2}(v(t) + v_0)t$$

We then substitute for v from Eq. 2.8 into Eq. 2.10 to obtain

$$x(t) - x_0 = \frac{1}{2}(v_0 + at + v_0)t \Longrightarrow$$

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \qquad 2.12$$

The Third Kinematic Equation for Constant Acceleration

If you get by eliminating the time t parameter given the two previously derived equations 2.8 and 2.12 then you will obtain third kinematic equation:

$$v^2(x) = v_0^2 + 2a(x - x_0) \qquad 2.13$$

The square of the speed of a particle at position x is equal to the sum of the initial speed squared plus twice the product of the acceleration and the distance from the initial position. Note that time t is absent from this third equation.

The SUPER–IMPORTANT kinematic equations of motion In ONE dimension with CONSTANT acceleration

Most important use: FREE FALL – the constant acceleration of the earth's gravity

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$v(t) \equiv$ **the speed of a particle at any time** t

$$v(t) = v_0 + at \tag{2.8}$$

$t \equiv$ the time parameter (in units of seconds for example)
$v_0 \equiv$ the initial (at $t = 0$) speed of the particle (in m/s for example)
$a \equiv$ the **constant** acceleration (in m/s^2 for example)

$x(t) \equiv$ **the position of a particle at any time** t

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \tag{2.12}$$

$x_0 \equiv$ the initial (at $t = 0$) position of the particle (in m for example)

$v(t) \equiv$ **the speed of a particle at any position** $x(t)$

$$v^2(x) = v_0^2 + 2a(x - x_0) \tag{2.13}$$

A not so super–important kinematic equation of motion,
(just an important kinematic equation of motion)

$x(t)$ **as a function of** $v(t)$**,** v_0 **and** t

$$x(t) = x_0 + \frac{1}{2}(v(t) + v_0)t \tag{2.14}$$

This equation can be derived from Equations 2.8 and Equation 2.12.

Notes on Textbook's Notation for Chapter 2 Kinematic Equations
The textbook writes Equation 2.8 $v(t) = v_0 + at$ as

$$v_x = v_{0x} + a_x t$$

You may find the double subscript somewhat confusing at first. The x means motion along one dimension, in this case the x axis. The "0" subscript means the initial value. I could put an x subscript in all of my notation for this Chapter 2, but I will leave that to Chapter 3 when we need both x and y subscripts. I also put $x(t)$ and $v(t)$ on the left sides to emphasize the dependence on time.

CONSTANT Acceleration Horizontal and Vertical Equations of Motion

Horizontal Motion Problems

Horizontal motion problems often involve cars accelerating or braking along a straight road, or the same for trains on a straight track, or airplanes in level flight. In the simplest problems the acceleration is the same (or zero) throught the motion. In more complex problems the acceleration may be one constant value for the first part of the motion, and then a different constant value for the second part of the motion The most complicated problems will have two objects in motion with possibly different acceleration constants.

However, you can solve all of these problems by applying the three kinematic equations which we have seen in the past two lectures. By convention for horizontal motion, we use the x axis as the direction of the motion. These three kinematic equations are again:

$$v(t) = v_0 + at \qquad 2.8$$

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \qquad 2.12$$

$$v^2(x) = v_0^2 + 2a(x - x_0) \qquad 2.13$$

In the description of the problem, look for the so-called initial conditions which are the values of the position or the speed when the time $t = 0$ seconds. These values represent the x_0 or the v_0 constants in the above equations.

Vertical Motion Free-Fall Problems

Free-Fall problems are those in the object is being accelerated by gravity, where the constant gravity acceleration value is

$$g = 9.8 \text{ m/s}^2$$

By convention we write the position coordinate as $y(t)$ and have the up direction as positive. With that convention in mind, the acceleration constant $a = -g$, and we can write the following comparable three equations of motion:

$$v(t) = v_0 - gt \qquad 2.8a$$

$$y(t) = y_0 + v_0 t - \frac{1}{2}gt^2 \qquad 2.12a$$

$$y^2(x) = v_0^2 - 2g(y - y_0) \qquad 2.13a$$

CONSTANT Acceleration Vertical Equations of Motion (continued)

Vertical Motion Free-Fall Problems

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$$y^2(x) = v_0^2 - 2g(y - y_0) \qquad 2.13a$$

You should be aware that for some vertical motion free-fall problems, the initial speed v_0 can be a positive value, meaning that the object is traveling upwards. However, the object is still being accelerated downwards. What that means is that upwards speed is constantly decreasing until it becomes zero. After that instant of time, the particle is traveling downwards and its downwards speed is becoming ever greater (larger negative values).

Problem Solving Strategy for Constant Acceleration Problems

Here are the rules for successfully solving accelerated motion problems using the three basic kinematics equations.

> Rule 1: *Make sure that all the kinematic quantities (distance, velocity, and acceleration) are in the same units: typically these will be in meters and seconds*

If some of the distance quantities are given in kilometers, for example, you may want to change them to meters or vice versa. Similarly if some of the time quantities are given in units other than seconds, as 5 meters/minute, then you almost always want to change those time units to seconds by dividing appropriately.

> Rule 2: *Choose a coordinate system.*

For one dimensional problems this will be either a horizontal (x) or a vertical (y) system. Make sure you choose your initial position and allow for negative values of the coordinate if appropriate

> Rule 3: *Make a (mental) list of what quantities you know, and what quantities you don't know*

This list is necessary because it will tell you which of the three equations can be used right away, and which cannot until more information is determined.

> Rule 4: *Construct a picture in your head of what is going on, and then select which equation of motion is appropriate.*

Before you start to play with the equations, you should have a clear idea in you mind of in which direction the particle is being accelerated, does it stop at anytime during the motion, and what is the relation between the velocity vector and the acceleration vector at different times.

> Rule 5: *Construct a motion diagram of the problem and list on that diagram which quantities you know and which you don't know.*

The motion diagram is perhaps the most important thing that you have to do. Unless you get that pictured right then it will be very hard to guess how to solve the problem.

Worked Example

A pitcher tosses a baseball straight up, with an initial speed of 12 m/s. How long does it take the ball to reach its highest point, and how high does it rise? How long will it take for the ball to reach 5.0 m above its release point?

Solution
First ask: what do we know?
We know the initial speed $v_0 = 12$ m/s
We know the acceleration $= g = 9.8$ m/s^2
We can define our coordinate system such that $y_0 = 0$ corresponding to the initial release point.
So we know: $y_0 = 0$, $v_0 = 12m/s$, and $g = 9.8$. So which equation do we use for the time to reach the highest point?

Call the time at the highest point t_1. At this moment the ball has zero velocity since it is at the highest point. So Eq. 2.8 becomes:

$$v(t_1) = 0 = v_0 - gt_1 = 12 - 9.8t_1 \Longrightarrow t_1 = \frac{12}{9.8} = 1.2 \text{ seconds}$$

For the position at the highest point, use Eq. 2.12

$$y(t_1) = y_0 + v_0 t_1 - \frac{1}{2} g t_1^2 = 0 + 12 \cdot 1.2 - 4.9 \cdot (1.2)^2 = 7.3 \text{ meters}$$

In order to find the time t at which the ball is at 5.0 meters, just use equation 2.12 again:

$$y(t) = 5.0 = y_0 + v_0 t - \frac{1}{2} g t^2 \Longrightarrow 4.9t^2 - 12t + 5.0 = 0$$

This is a *quadratic equation*[1] with the two time solutions being: $t = 0.53$ s and $t = 1.9$ s. Since we know "what goes up must come down", the first time value is for the upward rise of the ball, and the second time value is for the downward fall of the ball after it has reached its maximum height.

[1] You may recall that for a quadratic equation of the form $ax^2 + bx + c = 0$, there are two solutions given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

More difficult free-fall problem

A stone is thrown from the top of a building with an **initial velocity** of 20 m/s **straight up**. The building is **50 m high**, and the stone just misses the edge of the building on its way down. Determine

a) the **time** for the stone to reach its **maximum height**

b) the **maximum height**

c) the **time** needed for the stone to return to the **level of the thrower**

d) the **velocity** of the stone at the **time** of step c)

e) the **velocity** and the **position** of the stone at $t = 5$ seconds

f) the **velocity** of the stone just before it hits the **ground**

g) the total **time** that the stone is in the air

Solution

Ask yourself again: *What do I know?*

You know the initial speed $v_0 = ?$ You know the initial height $y_0 = ?$

You know, of course, the value of $g = 9.8$ m/s^2

Now start thinking

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a) What is the time to reach the maximum height? (same as previous example) This is not a plug–in question. There is no equation which gives the time for the maximum height directly. There is no equation which gives the maximum height directly either. But, you have to realize what it means for the particle to be at its maximum height. It means that the particle, *momentarily*, is neither rising nor falling. It has, *momentarily* stopped moving. Therefore, its velocity is 0. So we use the velocity equation.

$$v(t) = v_0 - gt = 20\frac{\text{m}}{\text{s}} - gt$$

Call the time for the maximum height t_{max}. Then

$$v(t = t_{max}) = 0 = 20\frac{\text{m}}{\text{s}} - 9.8\frac{\text{m}}{\text{s}^2}t_{max}$$

$$\Longrightarrow t_{max} = (20\frac{\text{m}}{\text{s}})/(9.8\frac{\text{m}}{\text{s}^2}) = 2.04 \text{ s}$$

More difficult free-fall problem (concluded)

b) What is the maximum height ? The maximum height, above ground level, is the value of the position function, $y(t)$, evaluated at $t = t_{max}$

$$y(t) = y_0 + v_0 t - \frac{1}{2} g t^2 = 50 + 20t - 4.9t^2$$

$$y(t = t_{max} = 2.04) = 50 + 20(2.04) - 4.9(2.04)^2 = 70.4 \text{ m}$$

---

c) What is the time needed to return to the level of the thrower? For this time value, use again the position equation $y(t) = y_0 + v_0 t - \frac{1}{2} g t^2$ and solve for the value of t such that $y(t) = y_0$. There is one obvious solution at $t = 0$ and there will also be a second, positive-value solution which is the time at which stone returns to its original 50 m height.

$$y(t) = y_0 + v_0 t - \frac{1}{2} g t^2 = 50 = 50 + 20t - 4.9t^2 \Longrightarrow 0 = t(20 - 4.9t)$$

$$\Longrightarrow t = 0 \quad \text{or} \quad t = 20/4.9 = 4.08 \text{ s}$$

Notice that 4.08 is twice the value of 2.04 so it takes the same time to fall from its maximum height to its initial position as it did to reach its maximum height.

---

d) For the velocity in step c) just use the vertical speed equation $v(t) = v_0 - gt$ with the second value of t determined in the previous step:

$$v(t = 4.08) = 20 - 9.8 \times 4.08 = -20.0 \text{ m/s}$$

---

e) For the velocity and position at $t = 5$ seconds one calculates:

$v(t = 5) = 20 - 9.8 \times 5 = -29.0$ m/s, $y(t = 5) = 50 + 20 \times 5 - 4.9 \times (5)^2 = 27.5$ meters

---

f) and g) For the velocity of the stone just before it hits the ground, one has to first solve for the total time (part g) and then use that time in the velocity equation. The total time t_T is obtained by knowing that at ground level the height is 0 m, so

$$y(t = t_T) = 0 = y_0 + v_0 t_T - \frac{1}{2} g t_T^2$$

$$50 + 20t_T - 4.9t_T^2 = 0$$

As before, this is quadratic equation with two solution values of t_T. One of these will be negative ($t_T = -1.75$ s, why?), and the other will be the correct, positive-value answer ($t_T = +5.83$ s) for which $v(t = 5.83) = -37.1$ m/s.

CHAPTER 3: Motion in Two Dimensions: Projectile Motion

This chapter considers motion of a particle in **two** dimensions. The two dimensions are specified as X and Y positions as a function of time.
The crucial thing to remember is that the X and Y motions are completely independent.
The kinematic equations of motion which you learned last time will be straightforwardly applied to the **separate** X and Y motions.

The definitions of **displacement**, **velocity**, and **acceleration** which you learned in one dimension are straightforwardly extended to two dimensions.
Displacement in two dimensions

$$\Delta\vec{r} \equiv \vec{r_f} - \vec{r_i} = (x_f - x_i)\,\hat{\mathbf{i}} + (y_f - y_i)\,\hat{\mathbf{j}}$$

Velocity in two dimensions
average velocity :

$$\overline{\mathbf{v_{if}}} \equiv \frac{\mathbf{\Delta r}}{\Delta t} = \frac{\mathbf{r_f} - \mathbf{r_i}}{t_f - t_i} = \frac{(x_f - x_i)}{t_f - t_i}\,\hat{\mathbf{i}} + \frac{(y_f - y_i)}{t_f - t_i}\,\hat{\mathbf{j}}$$

instantaneous velocity:

$$\mathbf{v(t)} \equiv \lim_{\Delta t \to 0} \frac{\mathbf{\Delta r}}{\Delta t} = \lim_{t_f \to t_i} \frac{(x_f - x_i)}{t_f - t_i}\,\hat{\mathbf{i}} + \frac{(y_f - y_i)}{t_f - t_i}\,\hat{\mathbf{j}} = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}$$

Acceleration in two dimensions
average acceleration :

$$\overline{\mathbf{a_{if}}} \equiv \frac{\mathbf{\Delta v}}{\Delta t} = \frac{\mathbf{v_f} - \mathbf{r_i}}{t_f - t_i} = \frac{(v_{xf} - v_{xi})}{t_f - t_i}\,\hat{\mathbf{i}} + \frac{(v_{yf} - v_{yi})}{t_f - t_i}\,\hat{\mathbf{j}}$$

instantaneous acceleration :

$$\mathbf{a(t)} \equiv \lim_{\Delta t \to 0} \frac{\mathbf{\Delta v}}{\Delta t} = \lim_{t_f \to t_i} \frac{(v_{xf} - v_{xi})}{t_f - t_i}\,\hat{\mathbf{i}} + \frac{(v_{yf} - v_{yi})}{t_f - t_i}\,\hat{\mathbf{j}} = a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}$$

Going from ONE to TWO Dimensions with Kinematics

In Lecture 2, we studied the motion of a particle in just **one dimension**. The concepts of velocity and acceleration were introduced. For the case of constant acceleration, the **kinematic** equations were derived so that at any instant of time, you could know the position, velocity, and acceleration of a particle in terms of the initial position and the initial velocity. Now the same thing will be done in **two dimensions**. It is important that you recall what you have learned in the one dimension case.

Review of one dimension, constant acceleration kinematics

In one dimension, all you need to know is the position and velocity at a given instant of time:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$
$$v_x(t) = v_{0x} + a_x t$$

Note that I have put a sub–script x in these above equations. For strictly one dimensional motion, such a sub–script is superfluous. However, it is useful in extending your knowledge of kinematics to two dimensions.

Extension of kinematics to two dimensions

In two dimensions, say X and Y, you need to know the position and velocity of particle as a function of time in two, **separate** coordinates. The particle, instead of being confined to travel only along a straight horizontal (or vertical) line, is now allowed to move in a plane. The extension of kinematics to two dimensions is very straightforward.

For the X coordinate: For the Y coordinate:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \qquad y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$
$$v_x(t) = v_{0x} + a_x t \qquad v_y(t) = v_{0y} + a_y t$$

Crucial feature for two dimensional kinematics

The absolutely **crucial** *feature for studying two dimensional kinematics with constant acceleration is to realize that the motion in the x direction and the motion in the y direction are completely independent.*

The x direction motion is governed by its own acceleration component a_x and its own initial conditions x_0 and v_{0x}. Similarly, the y direction motion is determined by its own acceleration component a_y and its own initial conditions y_0 and v_{0y}. The two sets of equations share a common time parameter t.