## REVIEW: Free Fall From Chapter 2

*Free Fall is the vertical motion of a particle where only the CONSTANT acceleration due to gravity is acting.*

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*If air resistance can be neglected, all bodies fall to the Earth's surface with the same acceleration.*

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Prior to the time of Galileo, it was believed that more massive objects fall to the Earth more quickly than do lighter objects. The most extreme example would be a feather and a cannonball.
Legend has it that Galileo climbed to the top of the leaning tower of PISA, and dropped two objects of very different masses from the top of the tower at the same time. For all practical purposes, all the objects not affected by air resistance arrived at the same time on the ground.

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## Thought Questions on Free Fall

Consider a ball thrown vertically upward.

- During what parts of the motion is the ball in *free fall*?
- What is the value of the acceleration vector at the maximum height of the motion when the ball has momentarily stopped?
- Is there a change in sign of the acceleration vector from when the ball is rising to when the ball is falling?

**REVIEW: The SUPER–IMPORTANT kinematic equations of motion**
**In ONE dimension with CONSTANT acceleration**

*Most important use: FREE FALL – constant acceleration of earth's gravity*

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$v(t) \equiv$ **the speed of a particle at any time** $t$

$$v(t) = v_0 + at \tag{2.8}$$

$t \equiv$ the time parameter (in units of seconds for example)
$v_0 \equiv$ the initial (at $t = 0$) speed of the particle (in m/s for example)
$a \equiv$ the **constant** acceleration (in m/s$^2$ for example)

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$x(t) \equiv$ **the position of a particle at any time** $t$

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \tag{2.12}$$

$x_0 \equiv$ the initial (at $t = 0$) position of the particle (in m for example)

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$v(t) \equiv$ **the speed of a particle at any position** $x(t)$

$$v^2(t) = v_0^2 + 2a(x(t) - x_0) \tag{2.13}$$

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**Special case of FREE FALL**

Constant acceleration downwards = $-g$ = -9.8 m/s$^2$ = -32 ft/s$^2$
By convention we write the position coordinate as $y(t)$ indicating the vertical direction (up is positive).

$$v(t) = v_0 - gt \qquad 2.8a$$

$$y(t) = y_0 + \frac{1}{2}(v(t) + v_0)t \qquad 2.12a$$

$$v^2(t) = v_0^2 - 2g(y(t) - y_0) \qquad 2.13a$$

**Constant Acceleration: Another Worked Example in One Dimension**

A sports car is traveling along a **straight road** at 140 km/hr. When the brakes are applied, it undergoes a uniform negative acceleration (**deceleration**) which slows it down to 70 km/hr in a distance of 200 m. a) What is the acceleration of the sports car? b) If it continues to accelerate at this rate how far will it travel while its speed decreases from 70 to 35 km/hr? c) How far does it travel as its speed changes from 140 km/hr to 0?

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First realize that this is a constant horizontal acceleration problem so that the three kinematic equations of motion apply.
Second, ask what do you know (and what don't you know)?
You know (for part a) the initial and final positions: $x_0 = 0$, $x_f = 200$ meters
You know (for part a) the initial and final speeds: $v_0 = 140$, $v_f = 70$ km/hr
You don't know the value of $a$ nor any of the times except $t_0 = 0$.

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*What is the acceleration of the sports car*

First review the three main kinematic equations to find which is suitable

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \quad \text{(don't know the times nor } a\text{)}$$

$$v(t) = v_0 + at \quad \text{(don't know the times nor } a\text{)}$$

$$v^2(x_f) = v_0^2 + 2a(x_f - x_0) \quad \text{(know all except } a \text{ so we choose this equation}$$

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To solve part a) just put in the values which you know

$$a = \frac{1}{2}\frac{v^2(x_f) - v_0^2}{x_f - x_0}$$

Now remember that the distance is in **meters** and the speeds are in **km/hr**. You must convert to common units.

## One Dimensional Car Deceleration Problem (continued)

To solve part a) just put in the values which you know

$$a = \frac{1}{2}\frac{v^2(x_f) - v_0^2}{x_f - x_0}$$

Now remember that the distance is in **meters** and the speeds are in **km/hr**. You must convert to common units.
We choose to have the speeds expressed in **m/s**, so 70 km/hr = $70 \cdot 10^3$m/3600 sec = 19.44 m/s and 140 km/hr = $140 \cdot 10^3$m/3600 sec = 38.89 m/s. Then

$$a = \frac{1}{2}\frac{(19.44)^2 - (38.89)^2}{200 - 0} = -2.84 \text{ m/s}^2$$

Note that there is a negative sign to this acceleration value, and this is as it should be for deceleration.

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To solve part b) use the same kinematic equation as for part a) only this time you know the value of the acceleration $a$:

$$v^2(x_f) = v_0^2 + 2a(x_f - x_0)$$

Here $v_f = 35$ km/hr, $v_0 = 70$ km/hr, and $(x_f - x_0)$ is the unknown

$$x_f - x_0 = \frac{v^2(x_f) - v_0^2}{2a}$$

As before you must convert the speeds into m/s. So 70 km/hr = 19.44 m/s and 35 km/hr = 9.72 m/s:

$$x_f - x_0 = \frac{(9.72)^2 - (19.44)^2}{2 \cdot (-2.836)} = 50.0 \text{ m}$$

## One Dimensional Car Deceleration Problem (concluded)

A sports car is traveling along a **straight road** at 140 km/hr. When the brakes are applied, it undergoes a uniform negative acceleration (**deceleration**) which slows it down to 70 km/hr in a distance of 200 m. a) What is the acceleration of the sports car? b) If it continues to accelerate at this rate how far will it travel while its speed decreases from 70 to 35 km/hr? c) How far does it travel as its speed changes from 140 km/hr to 0?

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To solve part c) use the same kinematic equation as for part a). In this case $x_f$ is unknown, $x_0 = 0$, and $v_f = 0$ m/s. The initial speed $v_0 = 140$ km/hr works out to be 38.89 m/s from before

$$x_f - x_0 = \frac{v^2(x_f) - v_0^2}{2a}$$

$$x_f = \frac{-v_0^2}{2a} = \frac{-(38.89)^2}{2 \cdot (-2.836)} = 266 \text{ m}$$

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*Extra question* How would you determine how much **time** it takes for the sports car to come to a complete stop from its 140 km/hr initial speed?

## CHAPTER 3: Motion in Two Dimensions: Projectile Motion

This chapter considers motion of a particle in **two** dimensions. The two dimensions are specified as $X$ and $Y$ positions as a function of time.
*The crucial thing to remember is that the $X$ and $Y$ motions are completely independent.*
The kinematic equations of motion which you learned last time will be straightforwardly applied to the **separate** $X$ and $Y$ motions.

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The definitions of **displacement**, **velocity**, and **acceleration** which you learned in one dimension are straightforwardly extended to two dimensions.
**Displacement in two dimensions**

$$\Delta\vec{r} \equiv \vec{r_f} - \vec{r_i} = (x_f - x_i)\,\hat{\mathbf{i}} + (y_f - y_i)\,\hat{\mathbf{j}}$$

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**Velocity in two dimensions**
**average velocity :**

$$\overline{\mathbf{v_{if}}} \equiv \frac{\mathbf{\Delta r}}{\Delta t} = \frac{\mathbf{r_f} - \mathbf{r_i}}{t_f - t_i} = \frac{(x_f - x_i)}{t_f - t_i}\,\hat{\mathbf{i}} + \frac{(y_f - y_i)}{t_f - t_i}\,\hat{\mathbf{j}}$$

**instantaneous velocity:**

$$\mathbf{v(t)} \equiv \lim_{\Delta t \to 0} \frac{\mathbf{\Delta r}}{\Delta t} = \lim_{t_f \to t_i} \frac{(x_f - x_i)}{t_f - t_i}\,\hat{\mathbf{i}} + \frac{(y_f - y_i)}{t_f - t_i}\,\hat{\mathbf{j}} = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}$$

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**Acceleration in two dimensions**
**average acceleration :**

$$\overline{\mathbf{a_{if}}} \equiv \frac{\mathbf{\Delta v}}{\Delta t} = \frac{\mathbf{v_f} - \mathbf{r_i}}{t_f - t_i} = \frac{(v_{xf} - v_{xi})}{t_f - t_i}\,\hat{\mathbf{i}} + \frac{(v_{yf} - v_{yi})}{t_f - t_i}\,\hat{\mathbf{j}}$$

**instantaneous acceleration :**

$$\mathbf{a(t)} \equiv \lim_{\Delta t \to 0} \frac{\mathbf{\Delta v}}{\Delta t} = \lim_{t_f \to t_i} \frac{(v_{xf} - v_{xi})}{t_f - t_i}\,\hat{\mathbf{i}} + \frac{(v_{yf} - v_{yi})}{t_f - t_i}\,\hat{\mathbf{j}} = a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}$$

## Going from ONE to TWO Dimensions with Kinematics

In Lectures 2 and 3, we studied the motion of a particle in just **one dimension**. The concepts of velocity and acceleration were introduced. For the case of constant acceleration, the **kinematic** equations were derived so that at any instant of time, you could know the position, velocity, and acceleration of a particle in terms of the initial position and the initial velocity. Now the same thing will be done in **two dimensions**. It is important that you recall what you have learned in the one dimension case.

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**Review of one dimension, constant acceleration kinematics**
In one dimension, all you need to know is the position and velocity at a given instant of time:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

$$v_x(t) = v_{0x} + a_x t$$

Note that I have put a sub–script $x$ in these above equations. For strictly one dimensional motion, such a sub–script is superfluous. However, it is useful in extending your knowledge of kinematics to two dimensions.

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**Extension of kinematics to two dimensions**
In two dimensions, say $X$ and $Y$, you need to know the position and velocity of particle as a function of time in two, **separate** coordinates. The particle, instead of being confined to travel only along a straight horizontal (or vertical) line, is now allowed to move in a plane. The extension of kinematics to two dimensions is very straightforward.

For the $X$ coordinate:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

$$v_x(t) = v_{0x} + a_x t$$

For the $Y$ coordinate:

$$y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

$$v_y(t) = v_{0y} + a_y t$$

## Worked Example of Two Dimensional Motion

A particle moves in the $xy$ plane with an $x$ component of acceleration $a_x = 4$ m/s$^2$. The particle starts from the origin at $t = 0$ with an initial velocity having an $x$ component of 20 m/s, and a $y$ component of $-15$ m/s. (There is no $y$ component of acceleration $\Longrightarrow a_y = 0$.)

a) What are the $x$ and $y$ components of the velocity vector as a function of time ?

b) What are the velocity and speed of the particle at $t = 5$ s ?

c) What are the $x$ and $y$ components of the position vector as a function of time ?

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a) Velocity kinematics equations:

$$v_x(t) = v_{0x} + a_x t = 20 \text{ m/s} + 4 \text{ m/s}^2 t$$

$$v_y(t) = v_{0y} + a_y t = -15 \text{ m/s}$$

$$\mathbf{v(t)} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}} = (20 \text{ m/s} + 4 \text{ m/s}^2 t)\,\hat{\mathbf{i}} - (15 \text{ m/s})\,\hat{\mathbf{j}}$$

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b) Velocity and speed at $t = 5$ s

Substitute for $t = 5$ s in the above equations for $\mathbf{v(t)}$

$$\mathbf{v(t = 5)} = (20 + 4(5))\,\hat{\mathbf{i}} - 15\,\hat{\mathbf{j}} = 40 \text{ m/s}\,\hat{\mathbf{i}} - 15 \text{ m/s}\,\hat{\mathbf{j}}$$

The *speed* is obtained by using, again, the Pythagorean theorem

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(40)^2 + (-15)^2} = 42.7 \text{ m/s}$$

$$\Theta_v = \tan^{-1}\frac{v_y}{v_x} = \frac{-15}{40} = -20.6^{\text{O}}$$

## Worked Example of Two Dimensional Motion (concluded)

A particle moves in the $xy$ plane with an $x$ component of acceleration $a_x = 4\ \text{m/s}^2$. The particle starts from the origin at $t = 0$ with an initial velocity having an $x$ component of 20 m/s, and a $y$ component of $-15$ m/s. (There is no $y$ component of acceleration $\Longrightarrow a_y = 0$.)

a) What are the $x$ and $y$ components of the velocity vector as a function of time ?

b) What are the velocity and speed of the particle at $t = 5$ s ?

c) What are the $x$ and $y$ components of the position vector as a function of time ?

c) The separate position functions $x(t)$ and $y(t)$, given that at $t = 0$ the values are $x_0 = 0$ and $y_0 = 0$:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \Longrightarrow x(t) = 20(\ \text{m/s})t + 2(\ \text{m/s}^2)t^2$$

$$y(t) = y_0 + v_{0x}t + \frac{1}{2}a_y t^2 \Longrightarrow y(t) = -15(\ \text{m/s})t$$

The general position vector $\mathbf{r}(\mathbf{t})$ is then given by

$$\mathbf{r}(\mathbf{t}) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} = (20(\ \text{m/s})t + 2(\ \text{m/s}^2)t^2)\,\hat{\mathbf{i}} + (-15(\ \text{m/s})t)\,\hat{\mathbf{j}}$$

We can determine the velocity vector $\vec{v}(t)$ by taking the time derivative of the position vector:

$$\mathbf{v}(\mathbf{t}) = \frac{d\mathbf{r}(\mathbf{t})}{dt} = (20 + 4t)\,\hat{\mathbf{i}} - 15\,\hat{\mathbf{j}}$$

## Special Case of PROJECTILE Motion

There is a special, very important case of **two dimensional** motion with constant acceleration. This is the case of projectile motion for which the vertical motion is governed by gravity, *and there is no acceleration in the horizontal direction.* So what can one say about the velocity in the horizontal direction?

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**Projectile Motion**, no horizontal acceleration, $a_x = 0$

$$v_x(t) = v_{0x} + a_x t \Longrightarrow v_x(t) = v_{0x}$$
$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \Longrightarrow x(t) = x_0 + v_{0x}t$$

general kinematic equations $\Longrightarrow$ specific projectile motion equations

In *projectile motion*, the horizontal velocity is constant and remains equal to the initial velocity. It is most important that you realize and remember that fact. By consequence, the distance traveled horizontally increases linearly with the duration of the time traveled.

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**Projectile Motion**, vertical acceleration $=a_y = -\mathbf{g}$

$$v_y(t) = v_{0y} - gt$$
$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

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**Initial Angle of Elevation for a Projectile**

A projectile will typically have a positive initial horizontal component of velocity $v_{0x}$ and usually a positive initial vertical component of velocity $v_{0y}$. These two components of the initial velocity determine the initial angle of elevation $\alpha_0$ above the horizontal direction for the projectile

$$\alpha_0 \equiv \tan^{-1}\frac{v_{0y}}{v_{0x}}$$

Equivalently, if one is given the initial speed $v_0$ (magnitude of the initial velocity) and the direction angle $\alpha_0$ of the initial velocity vector with respect to the horizontal, then the horizontal and the vertical components are determined by

$$v_{0x} = v_0 \cos\alpha_0 \quad \text{and} \quad v_{0y} = v_0 \sin\alpha_0 \qquad 3.19$$

### Special Case of PROJECTILE Motion: Common time parameter

In the projectile equations we have written separately the $x$ position as a function of time, $x(t)$, and the $y$ position as a function of time, $y(t)$. Now in each equation it is the same *time* that we are using. It is exactly the same tick on the clock or number on the digital watch that is being used. So, we may solved for the time variable from the $x(t)$ equation, and substitute that in the $y(t)$ equation.

$$x(t) = x_0 + v_{0x}t \Longrightarrow t = \frac{x}{v_{0x}}$$

Now substitute this in the $y(t)$ equation.

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2 = y_0 + v_{0y}(\frac{x}{v_{0x}}) - \frac{1}{2}g(\frac{x}{v_{0x}})^2$$

$$y(x) = y_0 + \frac{v_{0y}}{v_{0x}}x - \frac{1}{2}\frac{g}{v_{x0}^2}x^2$$

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**The Trajectory Equation**

The position $Y$ as a function of the position $X$ is given a special name: the **trajectory equation**

$$y(x) = y_0 + \frac{v_{0y}}{v_{0x}}x - \frac{1}{2}\frac{g}{v_{x0}^2}x^2$$

The motion of a **projectile**, in terms of the $x$ and the $y$ positions is a **parabola**. Notice, that on the left side, we have switched from writing $y(t)$ to $y(x)$ because on the right hand side we have eliminated the time coordinate $t$. If the initial position $y_0$ is taken to be 0, and then we recall that the ratio $v_{0y} = v_0 \sin\alpha_0$ and $v_{0x} = v_0 \cos\alpha_0$, then we can re-write the trajectory equation

$$y(x) = (\tan\alpha_0)x - \frac{g}{2v_0^2\cos^2\alpha_0}x^2 \qquad 3.27$$

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**The Range of a Projectile**

The **Range** of a projectile is the horizontal distance that the projectile will travel until it returns to its initial height (ground level in usual cases). From the Equation 3.27, it is easy to determine that the Range depends upon the initial speed and the initial angle of elevation according to the formula:

$$R(v_0, \alpha_0) = \frac{v_0^2 \sin 2\alpha_0}{g}$$ (What angle of elevation gives the maximum range?)

**Prototype Problem for Projectile Motion**

Suppose a cannon is fired at ground level with some initial velocity $\vec{v}$. That is, the cannonball exits the cannon with a speed $v$ at some angle $\alpha_0$ with respect to the horizontal axis. Describe the motion of the cannonball. Specifically

1) How high $h$ (vertical direction) does the cannonball go ?
2) How far $R$ horizontal direction does the cannonball go?
3) How much time $t_1$ does it take for the cannonball to reach its maximum height?
4) How much time does it take before falling back to the ground?

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**How high $h$ does the cannonball go?**

First we realize that the cannonball executes a parabolic path.
At the highest point of the trajectory we know that its vertical velocity component is 0. So we have

$$v_y(t_1) = 0 = v_{0y} - gt_1 \Longrightarrow t_1 = \frac{v_{0y}}{g} = \frac{v_0 \sin \alpha_0}{g}$$

where we take $t_1$ to be the time to reach the maximum height.
Note: $v_{0y} = v_0 \sin \alpha_0$ where $\alpha_0$ is the initial direction of the velocity vector.
Now we can use this time $t_1$ to substitute into the vertical position equation:

$$y(t_1) = h = y_0 + v_{0y}t_1 - \frac{1}{2}gt_1^2$$

$$h = 0 + v_{0y} \cdot \frac{v_{0y}}{g} - \frac{1}{2}g(\frac{v_{0y}}{g})^2 = \frac{v_{y0}^2}{2g} = \frac{v_0^2 \sin^2 \alpha_0}{2g}$$

So we have answered parts 3) and 1) above, and we should be able to quickly get the answer for part 4). *What is the answer to part 4)?*

## Circular Motion and Motion in a Curved Path

We have stated that acceleration is the time rate of change of vector velocity

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\vec{\Delta v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Now there are two ways that one can get a non-zero value for $\vec{\Delta v}$. The first, and most obvious, way is to have a change in the magnitude of $\vec{v}$. This is what we normally think of as acceleration: an increase (decrease deceleration) of speed. However, and this is not so obvious at first glance, we can also get a change in the velocity vector *even if the magnitude v does NOT change.* How is this possible. Simple. Just change the direction of the velocity vector $\vec{v}$. The change in the direction of $\vec{v}$, even if the magnitude v stays constant, produces a $\vec{\Delta v}$.

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## Motion in a Circle at Constant Speed

The simplest case of changing the direction of the velocity vector without changing the magnitude $v$ is to have motion in a circle of constant radius $r$ at a constant speed $v$. According to the above discussion, we must then have an acceleration. The magnitude of this acceleration is easily proven (page 87–89) to be:

$$a_c = \frac{v^2}{r} \tag{3.28}$$

Note that we have attached a subscript $c$ to the symbol for this acceleration. The reason to use this subscript is to indicate the *direction* which in this case is *along the radius towards the center of the circle.* This kind of acceleration is called *centripetal acceleration* meaning center-seeking acceleration.

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### Acceleration Has Two Components

In two dimensional motion acceleration has two *Cartesian* components $\vec{a} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}}$. The term Cartesian means using a rectangular coordinate system. However, in some cases it is more useful to think of the *tangential* and the *radial* components of the acceleration. It is still the same acceleration, but expressed in a different coordinate system.

$$\vec{a} = \vec{a}_t + \vec{a}_c$$

The *tangential component* is in the direction *tangent* to the path. This component of acceleration increases the magnitude of the velocity. The *radial* or *centripetal* component of the acceleration changes the direction of the velocity vector.

# Relative Velocity

## Different Coordinate Frames

As we saw on the last page, the same vector may be decomposed into its components in different coordinate systems. The idea of coordinate systems is an important one which we have been using so far without too much thought. However, the subject of coordinate systems (also called reference frames) is extremely important in Physics. In fact, it was by a study of how Physics is derived in different reference frames that Albert Einstein came up with his famous Theory of Relativity. For now we just show some simple examples.

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## Questions About Moving Reference Frames

Suppose you are in a train, and the window shades down and the track is very smooth, quiet, *and* straight. Can you tell that you are moving? For that matter, seated in the classroom, can you tell that the Earth is moving around the Sun? Suppose the train goes around a sharp curve, again with the window shades still shut. Can you tell whether this is happening?

Same train and on a straight track, but the window shades are up and you can see that you are moving very fast, say 100 miles/hour. You decide to stand in the aisle and jump straight up as high as you can. Where to you land in the aisle? What does your motion look like to a fellow passenger? What does your motion look like to someone looking through the window?

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## Velocities and Moving Reference Systems

Suppose that there is one coordinate system $O'$ moving at a constant velocity $\vec{V}_{OO'}$ with respect to another coordinate system $O$. Now a particle $P$ is observed to be moving in coordinate system $O$ with velocity $\vec{v}_{PO}$. In the $O'$ coordinate system an observer will see velocity $\vec{v}_{PO'}$. These three velocities are related by

$$\vec{v}_{PO'} = \vec{v}_{PO} - \vec{V}_{OO'} \qquad 3.22$$

The above equation is called the *Galilean Velocity Transformation.* It was named after Galileo, was was thought to be universally true up until Einstein found that it was not correct at velocities near the speed of light. However, you can use this velocity transformation equation quite well at normal speeds. In fact, there are interesting relative velocity problems such as flying in a cross-wind, or a boat crossing a river with a current which we can study. You already know how to solve these problems qualitatively, if not quantitatively, by your own experiences.