REVIEW: Going from ONE to TWO Dimensions with Kinematics

In Lecture 2, we studied the motion of a particle in just **one dimension**. The concepts of velocity and acceleration were introduced. For the case of constant acceleration, the **kinematic** equations were derived so that at any instant of time, you could know the position, velocity, and acceleration of a particle in terms of the initial position and the initial velocity. Now the same thing will be done in **two dimensions**. It is important that you recall what you have learned in the one dimension case.

Review of one dimension, constant acceleration kinematics

In one dimension, all you need to know is the position and velocity at a given instant of time:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

$$v_x(t) = v_{0x} + a_x t$$

Note that I have put a sub–script x in these above equations. For strictly one dimensional motion, such a sub–script is superfluous. However, it is useful in extending your knowledge of kinematics to two dimensions.

Extension of kinematics to two dimensions

In two dimensions, say X and Y, you need to know the position and velocity of particle as a function of time in two, **separate** coordinates. The particle, instead of being confined to travel only along a straight horizontal (or vertical) line, is now allowed to move in a plane. The extension of kinematics to two dimensions is very straightforward.

For the X coordinate:

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

$$v_x(t) = v_{0x} + a_x t$$

For the Y coordinate:

$$y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

$$v_y(t) = v_{0y} + a_y t$$

REVIEW: Special Case of PROJECTILE Motion

There is a special, very important case of **two dimensional** motion with constant acceleration. This is the case of projectile motion for which the vertical motion is governed by gravity, *and there is no acceleration in the horizontal direction.* So what can one say about the velocity in the horizontal direction?

Projectile Motion, no horizontal acceleration, $a_x = 0$

$$v_x(t) = v_{0x} + a_x t \Longrightarrow v_x(t) = v_{0x}$$
$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \Longrightarrow x(t) = x_0 + v_{0x}t$$

general kinematic equations $\Longrightarrow$ specific projectile motion equations

In *projectile motion*, the horizontal velocity is constant and remains equal to the initial velocity. It is most important that you realize and remember that fact. By consequence, the distance traveled horizontally increases linearly with the duration of the time traveled.

Projectile Motion, vertical acceleration $= a_y = -\mathbf{g}$

$$v_y(t) = v_{0y} - gt \quad \text{and} \quad y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

Initial Angle of Elevation for a Projectile
A projectile will typically have a positive initial horizontal component of velocity v_{0x} and usually a positive initial vertical component of velocity v_{0y}. These two components of the initial velocity determine the initial angle of elevation θ_0 above the horizontal direction for the projectile

$$\theta_0 \equiv \tan^{-1}\frac{v_{0y}}{v_{0x}}$$

Equivalently, if one is given the initial speed v_0 (magnitude of the initial velocity) and the direction angle $theta_0$ of the initial velocity vector with respect to the horizontal, then the horizontal and the vertical components are determined by

$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0 \quad \text{(The book uses } \alpha_0 \text{ instead of } \theta_0\text{)} \quad 3.19$$

REVIEW: Special Case of PROJECTILE Motion: Common time parameter

In the projectile equations we have written separately the x position as a function of time, $x(t)$, and the y position as a function of time, $y(t)$. Now in each equation it is the same *time* that we are using. It is exactly the same tick on the clock or number on the digital watch that is being used. So, we may solved for the time variable from the $x(t)$ equation, and substitute that in the $y(t)$ equation. We usually take $x_0 = 0$, which leads to

$$x(t) = v_{0x}t \Longrightarrow t = \frac{x}{v_{0x}}$$

Now substitute this in the $y(t)$ equation.

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2 = y_0 + v_{0y}(\frac{x}{v_{0x}}) - \frac{1}{2}g(\frac{x}{v_{0x}})^2$$

$$y(x) = y_0 + \frac{v_{0y}}{v_{0x}}x - \frac{1}{2}\frac{g}{v_{x0}^2}x^2 = y_0 + \tan\theta_0 x - \frac{1}{2}\frac{g}{v_0^2\cos^2\theta_0}x^2$$

where we have substituted $\tan\theta_0 = v_{0y}/v_{0x}$.

The Trajectory Equation

The position Y as a function of the position X is given a special name: the **trajectory equation**. If $y_0 = 0$ then we have

$$y(x) = \tan\theta_0 x - \frac{1}{2}\frac{g}{v_0^2\cos^2\theta_0}x^2$$

The motion of a **projectile**, in terms of the x and the y positions is a **parabola**. Notice, that on the left side, we have switched from writing $y(t)$ to $y(x)$ because on the right hand side we have eliminated the time coordinate t in favor of the position coordinate x.

The Range of a Projectile

The **Range** of a projectile is the horizontal distance that the projectile will travel until it returns to its initial height (ground level in usual cases). From the Equation 3.27, it is easy to determine that the Range depends upon the initial speed and the initial angle of elevation according to the formula:

$$R(v_0, \alpha_0) = \frac{v_0^2\sin 2\alpha_0}{g} \qquad \text{(What angle of elevation gives the maximum range?)}$$

Prototype Problem for Projectile Motion

Suppose a cannon is fired at ground level with some initial velocity $\vec{v}$. That is, the cannonball exits the cannon with a speed v at some angle θ with respect to the horizontal axis. Describe the motion of the cannonball. Specifically

1) How high h (vertical direction) does the cannonball go ?

2) How far R horizontal direction does the cannonball go?

3) How much time t_1 does it take for the cannonball to reach its maximum height?

4) How much time does it take before falling back to the ground?

How high h does the cannonball go?

First we realize that the cannonball executes a parabolic path. At the highest point of the trajectory we know that its vertical velocity component is 0. So we have

$$v_y(t_1) = 0 = v_{0y} - gt_1 \Longrightarrow t_1 = \frac{v_{0y}}{g} = \frac{v_0 \sin\theta_0}{g}$$

where we take t_1 to be the time to reach the maximum height.
Note: $v_{0y} = v_0 \sin\theta_0$ where θ_0 is the initial direction of the velocity vector.
Now we can use this time t_1 to substitute into the vertical position equation:

$$y(t_1) = h = y_0 + v_{0y}t_1 - \frac{1}{2}gt_1^2$$

$$h = 0 + v_{0y} \cdot \frac{v_{0y}}{g} - \frac{1}{2}g(\frac{v_{0y}}{g})^2 = \frac{v_{y0}^2}{2g} = \frac{v_0^2 \sin^2\theta_0}{2g}$$

So we have answered parts 3) and 1) above, and we should be able to quickly get the answer for part 4). *What is the answer to part 4)?*

Projectile Equations: Initial Position at Origin

The **trajectory equation** was previously shown to be:

$$y(x) = y_0 + \frac{v_{0y}}{v_{0x}}x - \frac{1}{2}\frac{g}{v_{x0}^2}x^2$$

In terms of the cannonball problem, where we specify v and θ, instead of v_{0x} and v_{0y}, and have $x_0 = 0 = y_0$ this is easily transformed to be:

$$y(x) = (\tan\theta_0)x - \Big(\frac{g}{2v_0^2\cos^2\theta_0}\Big)x^2 \qquad (3.27)$$

The text book uses α_0 instead of θ_0 to represent the initial angle of the projectile.

Horizontal Range

A distance of interest is the **Horizontal Range** which the text symbolizes with the letter R. The horizontal range is the x distance which the projectile travels before returning to the ground level. The solution for R can be obtained by solving trajectory equation for $y(R) = 0$:

$$y(R) = 0 = (\tan\theta_0)R - \Big(\frac{g}{2v_0^2\cos^2\theta_0}\Big)R^2$$

One solution is $R = 0$ (why?), and the other solution is given by

$$R = \frac{v_0^2\sin 2\theta_0}{g}$$

Note that the maximum value of the Range occurs at $\theta_0 = 45^{\text{O}}$, and is given by:

$$R_{max}(\theta_0 = 45^{\text{O}}) = \frac{v_0^2}{g}$$

You should also convince yourself that different values θ_0 which are symmetric about $\theta_0 = 45^{\text{O}}$ will give the same value for the Range.

Worked Example: "Shoot–the–Monkey" demonstration

The hypothetical premise of this ecologically-incorrect example is that a hunter spies a monkey hanging from a tree branch. The hunter knows that when he fires his rifle, the monkey will drop instantaneously from the tree. Where should the hunter aim his rifle: 1) above the monkey, 2) at the monkey, or 3) below the monkey?

How do you quantify the fact that an "intercept" has occurred, in other words the projectile fired from the gun hits the dropping target ?

An intercept will occur if *at the same time* the projectile and the target are at exactly the same coordinates (x, y)

1) Since the target is just dropping, its horizontal position remains the same at all time: $x \equiv x_T$

2) Now calculate how long it takes the projectile to reach the $x = x_T$

$$x(t) = v_{0x}t = (v_0 \cos\theta_0)t$$

$$\Longrightarrow t(intercept) \equiv t_I = \frac{x_T}{v_0 \cos\theta_0}$$

3) Now calculate the vertical position y_P where the projectile is at $t = t_I$

$$y_P(t_I) = v_{0y}t_I - \frac{1}{2}gt_I^2 = (v_0 \sin\theta_0)(\frac{x_T}{v_0 \cos\theta_0}) - \frac{1}{2}g(\frac{x_T}{v_0 \cos\theta_0})^2$$

$$y_P(t_I) = x_T \tan\theta_0 - \frac{1}{2}g\Big(\frac{x_T}{v_0 \cos\theta_0}\Big)^2$$

4) For the dropping target, its initial height $y_0 = x_T \tan\theta_0$, its initial velocity is 0, and so its position at $t = t_I$ is given by:

$$y_T(t_I) = y_0 - \frac{1}{2}gt^2 = x_T \tan\theta_0 - \frac{1}{2}g\Big(\frac{x_T}{v_0 \cos\theta_0}\Big)^2$$

So both the target and the projectile meet at the same (x, y) coordinates simultaneously. Notice that this result is independent of v_0.

Circular Motion and Motion in a Curved Path

We have stated that acceleration is the time rate of change of vector velocity

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\vec{\Delta v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Now there are two ways that one can get a non-zero value for $\vec{\Delta v}$. The first, and most obvious, way is to have a change in the magnitude of $\vec{v}$. This is what we normally think of as acceleration: an increase (decrease deceleration) of speed. However, and this is not so obvious at first glance, we can also get a change in the velocity vector *even if the magnitude v does NOT change.* How is this possible. Simple. Just change the direction of the velocity vector $\vec{v}$. The change in the direction of $\vec{v}$, even if the magnitude v stays constant, produces a $\vec{\Delta v}$.

Motion in a Circle at Constant Speed

The simplest case of changing the direction of the velocity vector without changing the magnitude v is to have motion in a circle of constant radius r at a constant speed v. According to the above discussion, we must then have an acceleration. The magnitude of this acceleration is easily proven (page 87–89) to be:

$$a_c = \frac{v^2}{r} \quad \text{(The book uses “radial” } a_{\text{rad}}\text{)} \tag{3.28}$$

Note that we have attached a subscript c to the symbol for this acceleration. The reason to use this subscript is to indicate the *direction* which in this case is *along the radius towards the center of the circle.* This kind of acceleration is called *centripetal(or radial) acceleration* meaning center-seeking acceleration.

Acceleration Has Two Components

In two dimensional motion acceleration has two *Cartesian* components $\vec{a} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}}$. The term Cartesian means using a rectangular coordinate system. However, in some cases it is more useful to think of the *tangential* and the *radial* components of the acceleration. It is still the same acceleration, but expressed in a different coordinate system.

$$\vec{a} = \vec{a}_t + \vec{a}_c$$

The *tangential component* is in the direction *tangent* to the path. This component of acceleration increases the magnitude of the velocity. The *radial* or *centripetal* component of the acceleration changes the direction of the velocity vector.

Relative Velocity

Different Coordinate Frames

As we saw on the last page, the same vector may be decomposed into its components in different coordinate systems. The idea of coordinate systems is an important one which we have been using so far without too much thought. However, the subject of coordinate systems (also called reference frames) is extremely important in Physics. In fact, it was by a study of how Physics is derived in different reference frames that Albert Einstein came up with his famous Theory of Relativity.

Questions About Moving Reference Frames

Suppose you are in a train, and the window shades down and the track is very smooth, quiet, *and* straight. Can you tell that you are moving? For that matter, seated in the classroom, can you tell that the Earth is moving around the Sun? Suppose the train goes around a sharp curve, again with the window shades still shut. Can you tell whether this is happening?

Same train and on a straight track, but the window shades are up and you can see that you are moving very fast, say 100 miles/hour. You decide to stand in the aisle and jump straight up as high as you can. Where to you land in the aisle? What does your motion look like to a fellow passenger? What does your motion look like to someone looking through the window?

Velocities and Moving Reference Systems

Suppose that there is one coordinate system O' moving at a constant velocity $\vec{V}_{OO'}$ with respect to another coordinate system O. Now a particle P is observed to be moving in coordinate system O with velocity $\vec{v}_{PO}$. In the O' coordinate system an observer will see velocity $\vec{v}_{PO'}$. These three velocities are related by

$$\vec{v}_{PO'} = \vec{v}_{PO} - \vec{V}_{OO'} \qquad 3.36$$

The above equation is called the *Galilean Velocity Transformation.* It was named after Galileo, was was thought to be universally true up until Einstein found that it was not correct at velocities near the speed of light. However, you can use this velocity transformation equation quite well at normal speeds. In fact, there are interesting relative velocity problems such as flying in a cross-wind, or a boat crossing a river with a current which we can study. You already know how to solve these problems qualitatively, if not quantitatively, by your own experiences.

Worked Example of Centripetal Acceleration

Problem Statement A satellite is in circular orbit at 150 kilometers height (h) above the Earth's surface, where the Earth has a radius R_E of 6380 kilometers. The satellite orbits the Earth once every 90 minutes. What is the magnitude of the centripetal acceleration experienced by the satellite?

Solution In order to calculate centripetal acceleration $a_c = v^2/r$, we need to know the speed v and the radius of the orbit which in this case is $r = R_E + h =$ 6530 kilometers. The speed is the circumference of the orbit ($= 2\pi r$) divided by the time of the orbit which is $90 \times 60 = 5400$ seconds. Hence the speed is

$$v = \frac{2\pi r}{t} = \frac{2 \times 3.14159 \times 6.530 \times 10^6}{5400} = 7600 \text{ m/s}$$

Therefore the centripetal acceleration is

$$a_c = \frac{v^2}{r} = \frac{(7600)^2}{6.530 \times 10^6} = 8.8 \text{ m/s}^2$$

Are you curious why this centripetal acceleration is so close to $g = 9.8$ m/s^2? The answer will be given partially in the next two chapters.

Worked Example of Relative Velocity

Problem Statement A boat can travel at a speed of 4 meters/second in a river. The river itself has a current with a speed of 3 meters/second in the direction North. If the captain of the boat points the boat in the direction East, what will be the actual velocity of the boat with respect to the Earth's surface?

Problem Solution It is essential that you draw velocity vectors in order to solve such problems. You can easily visualize that the boat will be carried along in the Northerly direction by the river's current as well as making its way East under its own power. With the numbers given, the total magnitude of the total velocity is 5 meters/second. The direction is given by $\theta = \tan^{-1}(3/4) = 37^{\text{o}}$, North of East.

Now suppose that the river is 200 meters wide West to East. How far downstream from its initial position on the West bank will the boat arrive on the East bank? What quantity do you need to calculate first in order to solve this problem?

Chapter 4: Newton's Three Laws of Motion

First Law: The Law of Inertia

An object at rest will remain at rest unless and until acted upon by an external force.

An object moving at constant velocity will continue to move at constant velocity unless and until acted upon by an external force.
These two statements are the same when one considers that whether or not an object has velocity depends upon one's frame of reference.

Second Law: Relation between Force and Acceleration

An object will experience an acceleration **a** *in direct proportion to the force* **F** *exerted on the object. The constant of the proportionality is the object's mass* **m**.

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \tag{4.7}$$

If there is no force then there must be no acceleration. If there is no acceleration then there must be no force. If there is force then there must be acceleration. If there is acceleration then there must be a force.

Third Law: Action and Reaction Forces

Every (action) force which exists has an equal an opposite (reaction) force, but the action-reaction forces NEVER act on the same body.

A force $\vec{\mathbf{F}_{\mathbf{AB}}}$ is exerted on body A by body B. By Newton's Third Law there must also be a force $\vec{\mathbf{F}_{\mathbf{BA}}}$ which is exerted on body B by body A.

$$\vec{\mathbf{F}_{\mathbf{AB}}} = -\vec{\mathbf{F}_{\mathbf{BA}}} \tag{4.11}$$

"It takes two to tango" could be just another way of stating Newton's Third Law. In other words, all forces are the result of two bodies (particles) acting on one another.

Insight into Newton's First Law of Motion

First Law: The Law of Inertia

An object at rest will remain at rest unless and until acted upon by an external force.
An object moving at constant velocity will continue to move at constant velocity unless and until acted upon by an external force.

This may be the most difficult law of all to grasp. In fact, for at least 2,000 years, until the time of Galileo and Newton, the law was simply not recognized by humankind.

The View of Aristotle

The view which pervaded human thought until the 1600s was that objects were "naturally" at rest on the surface of the Earth. In order to keep an object moving, some force was necessary, although the word "force" would not have been used. Once the force was taken away, then the object would slow down and return to its "natural" state of rest.
Of course, there were certain objects such as the Sun, the Moon, the stars and the planets, which appeared to be in a perpetual state of motion, unaided by external force. Since this motion was not "natural", then the objects had to be "supernatural" or gods.

The View of Newton (Galileo and others of the time)

The present scientific view of motion is called *Newtonian Mechanics*, after Sir Issac Newton. It was Newton who discerned the connection between motion and force, and he was able to dispel the notion of a "natural" state of rest. His First Law was truly a revolution, and we can best appreciate it through the air track demonstration shown in class.

Astronomy and Newton's First Law of Motion

We conclude the study of Newton's First Law by returning to the observations of the seemingly perpetual motion of astronomical objects such as the Sun, the Moon, and the planets.
Here again is the First Law:
An object at rest will remain at rest unless and until acted upon by an external force.
An object moving at constant velocity will continue to move at constant velocity unless and until acted upon by an external force.

Now we ask these questions relative to the First Law:

1) Is the motion of the planets consistent with the First Law?

2) Specifically, are all of these objects moving with constant velocity, and how does one define constant velocity?

3) If a given astronomical object such as the Moon is not moving with constant velocity, what inference can we derive from the First Law

With the answer to this last question, we (as was Newton) are drawn into the discussion of the Second Law.

CHAPTER 4: Newton's Second Law of Motion

The fundamental equation of mechanics is Newton's Second Law of Motion:

$$\mathbf{F} = m\mathbf{a} \tag{4.7}$$

A **FORCE** acting on an object with mass m will produce an acceleration $\mathbf{a}$. Note that this is a vector equation, so it actually represents three separate equations for the X, Y, and Z components of the force and the acceleration.

More than one force acting on an object

If there is more than one force acting on an object, the left hand side of the Newton's second law is simply replaced by the vector sum of the forces acting on the body:

$$\Sigma_{i=1}^{N}\mathbf{F}_i = m\mathbf{a} \tag{4.8a}$$

$$\Longrightarrow \Sigma_{i=1}^{N}F_{i_x} = ma_x \qquad \Sigma_{i=1}^{N}F_{i_y} = ma_y \qquad \Sigma_{i=1}^{N}F_{i_z} = ma_z \tag{4.8b}$$

It is very important that you realize that this equation applies to the *forces acting on the object*. The most difficult thing for most students is to recognize all, and only all, the forces which act on a particular object. Sometimes students will leave out certain forces, and other times students will include forces which are not acting on the object in question. The secret is to focus your attention on the object in question.

ACCELERATED MOTION

Accelerated motion occurs when the left hand side of Eq. 4.7 is not zero. Then the object *must* have an acceleration. Many of the problems you will have to work out will involved computing the acceleration of an object when the object experiences certain forces. Typical cases will include the force of gravity (weight), the *normal forces* exerted by supporting walls and floors, and the *static* or *kinetic* frictional forces.

Worked Example: Newton's Second Law of Motion

A hockey puck with a mass of 0.3 kg *slides* on the horizontal frictionless surface of an ice rink. Two forces act on the puck. The force $\mathbf{F_1}$ has a magnitude of 5 N and acts at an angle of $-20^{\rm O}$, and the force $\mathbf{F_2}$ has a magnitude of 8 N and acts at an angle of $+60^{\rm O}$with respect to the x axis. Determine the acceleration of the puck.

Solution

This is a straightforward application of Newton's second law

$$\mathbf{F} = m\mathbf{a} \qquad \Longrightarrow \mathbf{a} = \frac{\mathbf{F}}{m} \tag{4.7}$$

We have to compute the *net* or resultant force acting on the mass of the puck. The acceleration vector $\mathbf{a}$ will be in the same direction as $\mathbf{F}$, and with a magnitude equal to the magnitude F divided by m.

Force	Magnitude (N)	Angle (deg.)	X Comp. (N)	Y Comp. (N)
$\mathbf{F_1}$	5.0	-20.0	$+4.70$	-1.71
$\mathbf{F_2}$	8.0	$+60.0$	$+4.00$	$+6.93$
$\mathbf{F_{net}}$	$F_{net} =$	$\theta_{net} =$	$+8.70$	5.22

So, by the usual methods of analytic vector addition, we find

$$F_{net} = \sqrt{(8.70)^2 + (5.22)^2} = 10.15 \text{ N}$$

$$\theta_{net} = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{5.22}{8.70} = 31.0^{\rm O}$$

Now to compute the acceleration

$$a = \frac{F_{net}}{m} = \frac{10.15}{0.3} = 33.8 \text{ m/s}^2$$

The direction of $\mathbf{a}$ is exactly the same as that of $\mathbf{F_{net}}$: $\theta_{\mathbf{a}} = 31.0^{\rm O}$.