

### REVIEW: Centripetal Force and Circular Motion

*Standard scientific literacy question:* A object is attached to a string and the object is traveling in a horizontal circle. The string breaks. What happens to the object ? Does it go in a curved horizontal path from the point at which it broke free ?

*Answer:* — **absolutely not**. If the object breaks free, its horizontal path is a **straight line**. The only way for the object to go in a curved or circular path in the horizontal plane is to have a force acting perpendicular to its velocity in the horizontal plane.

The force which makes an object go in a circle is called the **centripetal force**. Centripetal force is required to produce centripetal acceleration. You cannot have one without the other. If you see circular motion you can be sure that there is a centripetal force being produced somewhere, somehow.

$$F_r = ma_r = \frac{mv^2}{r} \quad \text{or} \quad F_c = ma_c = \frac{mv^2}{r}$$

The subscripts  $c$  for centripetal or  $r$  for radial can be used interchangeably.

### Centripetal Force Worked Example

A 1500 kg car is moving on a horizontal road which has a curve (part of a circle) with a radius of curvature of 35 m. The coefficient of *static friction* between the tires and the road is 0.50. What is the maximum speed at which the car can make this curve?

It is important that you realize that there must be a centripetal force exerted on the car (actually on the tires) by the road's surface in this case. It is this force which changes the direction of the car's velocity vector. Now the maximum static friction force is given by

$$f_s^{max} = \mu_s N = \mu_s Mg = \frac{Mv_{max}^2}{R}$$

$$\implies v_{max} = \sqrt{\mu_s g R} = \sqrt{0.5 \cdot 9.8 \cdot 35} = 13.1 \text{ m/s} \quad (\text{independent of mass!})$$

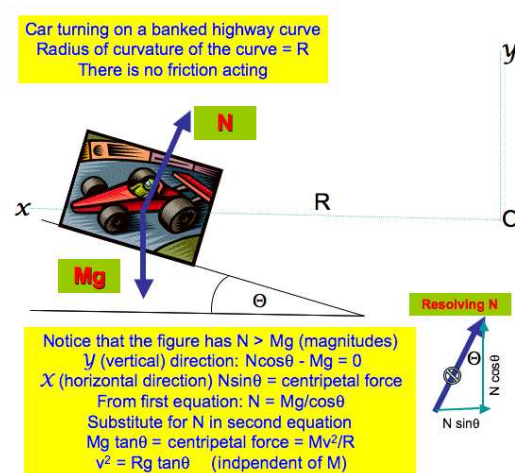
Do you understand why static friction is being used, and not kinetic friction?

*Answer:* Believe or not, the contact point of the tire on the road surface is *instantaneously at rest* if the car is not skidding. We learn about that when we discuss rotational motion.

## Chapter 5: Centripetal Force and Improved Highway Engineering

In the previous example, the centripetal force which enabled the car to make the curve came just from the force of static friction. If the force of static friction is reduced, because of wet or icy roads, then the maximum safe speed for making the turn may be drastically reduced.

In order to rely less on friction, highway engineers came up with the idea of *banking* the curves. The road surface, on a curve, becomes effectively an inclined plane. The normal reaction force of the road surface is no longer purely vertical, but it acquires a horizontal component. That horizontal component functions as the centripetal force which allows the car to turn in a circle.



In *banked* curves, it is possible for friction to be completely absent and still the car can make the turn. The centripetal force is given by the horizontal component of the normal force

$$F_{\text{centripetal}} = N \sin \theta = \frac{Mv^2}{R} \quad \text{and} \quad N \cos \theta = Mg$$

$$F_{\text{centripetal}} = \frac{Mg}{\cos \theta} \sin \theta = \frac{Mv^2}{R} \implies v = \sqrt{Rg \tan \theta} \quad (\text{independent of } M!)$$

Note again that the maximum speed here is independent of the mass of the car and depends just on the radius  $R$ , the banking angle  $\theta$ , and the gravitational acceleration constant  $g$

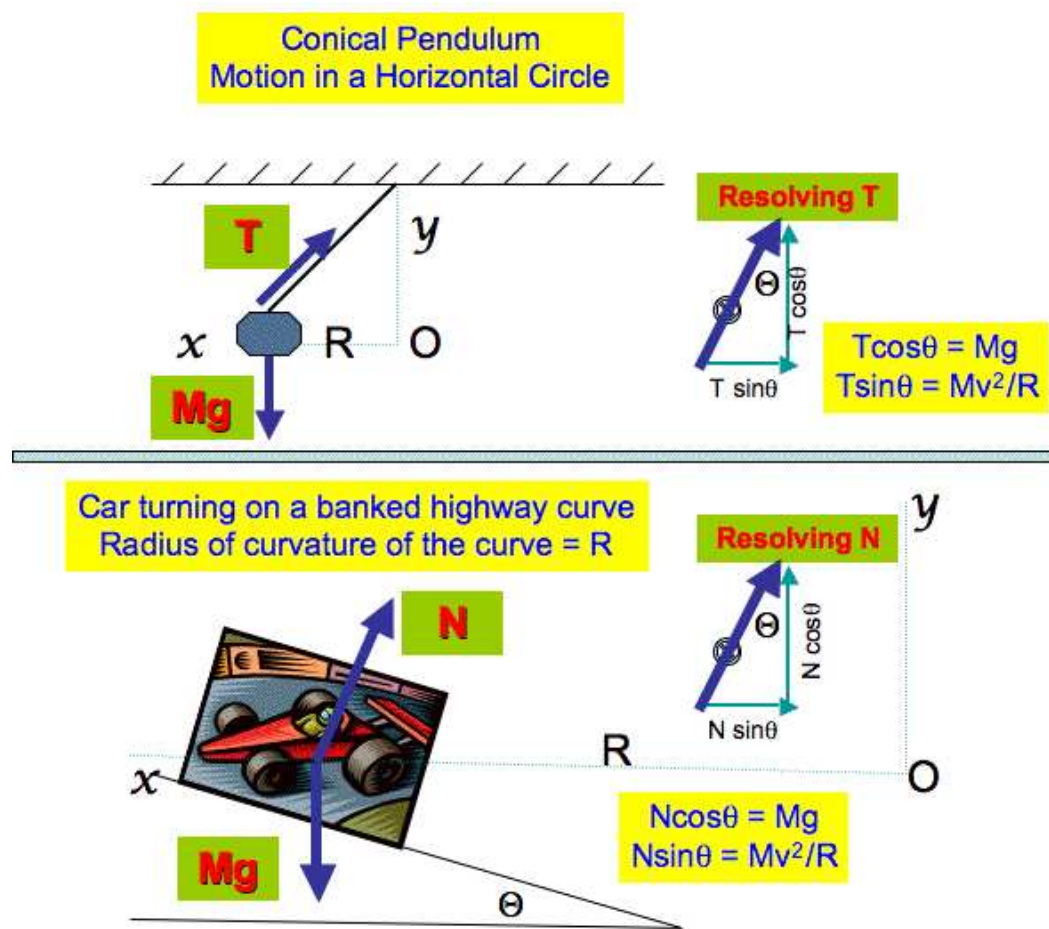
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For home study: suppose in addition to a banked angle, there was a coefficient of static friction  $\mu_s$ . How would you solve for the maximum speed in that case ?

### Chapter 5: Worked Example using CENTRIPETAL FORCE

Another example of centripetal motion is the *conical pendulum*

An object of mass  $m$  is suspended vertically from a string, and the object moves at constant speed  $v$  in a horizontal circle of radius  $R$ . The string makes an angle  $\theta$  with respect to the vertical direction. What is the speed of  $m$  (in terms of  $\theta$ ,  $g$ , and  $R$ ) ?



**Chapter 5: Worked Example using CENTRIPETAL FORCE**

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*Solution:* The key in circular motion problems is to find the force component which is acting centripetally. This force component must be directed at the center of the circle and perpendicular to the instantaneous velocity vector of the object. In this case there are two forces: 1) the weight vector acting vertically down, and 2) the tension vector of the string acting at some angle  $\theta$  with respect to the horizontal. Next, analyze the acceleration of the object. In which direction(s)  $x$  (horizontal) or  $y$  (vertical) does the particle have an acceleration ? If the acceleration is 0, then the net force acting in that direction must also be 0. For the conical pendulum, the object has no acceleration in the vertical direction. Therefore, the net force in the  $y$  direction is 0:

$$F_y = 0 = T_y - W = T \cos \theta - Mg \implies T \cos \theta = Mg$$

In the horizontal ( $x$ ) direction, *there is an acceleration !* This is the centripetal acceleration because the horizontal velocity is changing continuously. Therefore, there is a net force in the horizontal direction

$$F_x = T_x = Ma_x \implies T \sin \theta = \frac{Mv^2}{R}$$

Take these two equations and divide the second by the first:

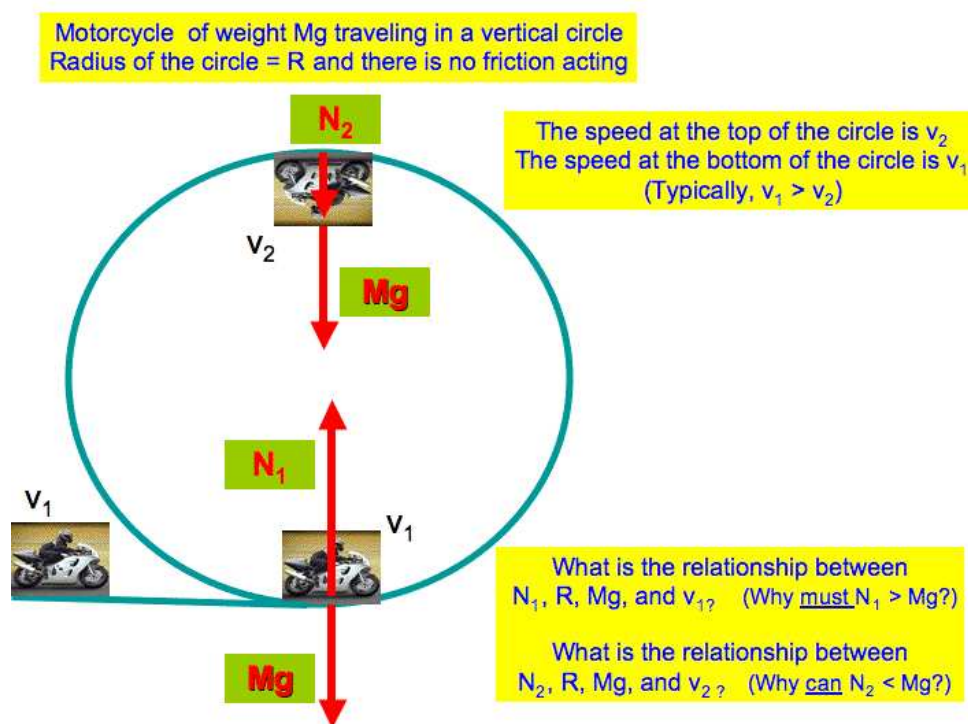
$$\frac{T \sin \theta}{T \cos \theta} = \frac{Mv^2}{RMg}$$

$$\tan \theta = \frac{v^2}{Rg} \implies v = \sqrt{Rg \tan \theta} \quad (\text{independent of } M!)$$

Notice that the conical pendulum result is the same as the banked highway result. The only difference in the two problems is that a normal force  $N$  is changed into a tension force  $T$ . The free-body diagrams are otherwise identical.

### Chapter 5: Centripetal FORCE in a VERTICAL Circle

So far all these examples of centripetal force have been for motion in a horizontal circle. The last example for this chapter will be for motion in a *vertical circle*. A motorcycle stuntman (Allo “Dare Devil” Diavolo was the first to do this trick in 1901) rides a motorcycle around a “loop-the-loop” track. What speed must he have had at the top of the loop (where he is upside-down) in order not to fall off the loop, where the loop has a radius  $R$ ? (Friction is not a factor.)



Since he is moving in a circle, he must have a centripetal force. At the top of the loop, that force is pointing straight down. The question you must ask yourself is where is that force coming from. The answer depends on how fast he is moving. In general, if his speed at the top of the loop is  $v$ , then there **must** be a centripetal force  $F_c$  given by:

$$F_c = \frac{Mv^2}{R}$$

Now, as to where this force is coming from, the easiest answer is gravity. There will **always** be a gravitational or weight force  $w = Mg$ .

Suppose that the weight force is bigger than the centripetal force. In other words suppose that the speed  $v$  is not so great. In that case, only part of the weight force  $w$  is “used up” by the centripetal force. So the motorcycle will fall off the track.

**Chapter 5: Centripetal FORCE in a VERTICAL Circle**

$$F_c = \frac{Mv^2}{R}$$

$$w = Mg$$

Suppose instead that the weight force is less than the centripetal force. Then the weight force will be insufficient to provide enough centripetal force. Where does the rest of the centripetal force come from? It comes from the track, specifically the **Normal Force** of the track on the motorcycle.

The limiting case is for the weight force to be exactly equal to the centripetal force. In that case, there is no need for a normal force but the motorcycle does not fall off the track.

$$F_c = w \quad \text{and} \quad \frac{Mv^2}{R} = Mg$$
$$\implies v = \sqrt{gR}$$

Note that the minimum speed  $v$  does NOT depend on the mass  $M$ . So it does not depend on how heavy the man is, nor how heavy the motorcycle is. It only depends upon  $g$  and the radius  $R$ .

## Chapter 6: Work and Kinetic Energy

The concept of **WORK** has a very precise definition in physics. **Work** is a physical quantity produced when a **Force** moves an object through a **Displacement** in the same direction as the force. Work is a *scalar* quantity.

In physics, in order to get the **Work** done, there must be a force and there must be motion caused by that force. If the force  $\vec{\mathbf{F}}$  is constant, and the displacement  $\vec{\mathbf{s}}$  is exactly in the same direction as the direction of the force, then the magnitude of the work is easy to calculate:

$$W = Fs \quad \text{constant force } \vec{\mathbf{F}} \text{ has the same direction as displacement } \vec{\mathbf{s}} \quad 6.1$$

### Work if the force is not in the same direction as the displacement

If the constant force  $\vec{\mathbf{F}}$  is not in the same direction as the displacement vector  $\vec{\mathbf{s}}$ , then the magnitude of the work done is given by

$$W = Fs \cos \theta \quad \text{constant } \vec{\mathbf{F}} \text{ separated by angle } \theta \text{ from } \vec{\mathbf{s}} \quad (6.2)$$

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} \quad \text{Work is the "dot product" of the vectors } \vec{\mathbf{F}} \text{ and } \vec{\mathbf{s}} \quad (6.3)$$

### Kinetic Energy    The Energy of Moving Objects

Consider the following logical progression: When a force acts on an object of mass  $m$  for a specific period of time causing the object to move, then work  $W$  is done by the force. If the object has a force acting on it, then it will accelerate. If the object was initially at rest, then after the force stops acting the object will have a speed  $v$ . We can define the **Kinetic Energy**  $K$  of the object to be

$$K = \frac{1}{2}mv^2 \quad (\text{VERY IMPORTANT EQUATION!}) \quad (6.5)$$

*The kinetic energy of an object moving with speed  $v$  is equal to the work done by the force which made the object move from rest to the speed  $v$ .*

In physics **Power** is defined as the **rate** at which work is done by a force. For a constant (time-independent) force  $\vec{\mathbf{F}}$ , one has the following *instantaneous* rate of work for a changing displacement vector  $\vec{\mathbf{s}}$

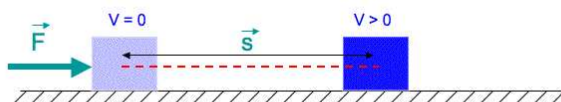
$$P(t) = \frac{dW}{dt} = \frac{d}{dt}(\vec{\mathbf{F}} \cdot \vec{\mathbf{s}}) = \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{s}}}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}(t) \quad (6.19)$$

## Simple Example of WORK in One Dimension

The rule to remember about work is:

*If it moves where you push it, then you're getting the work done.*

Work in One Dimension with the Displacement in the Same Direction as the Force  
 Constant Force  $\vec{F}$  Acts Continuously over a Displacement  $\vec{s}$   
 (There is no friction acting in this example.)



Simplest Case  
 Work = Force times Distance =  $Fs$   
 Unit of Work = Newton-meter = Joule

Consider an example of a box resting on a frictionless table. A horizontal force  $\vec{F}$  is applied to that box. By Newton's second law, the box will acquire an acceleration. After the box moves a displacement  $\vec{s}$  along the table, the force  $\vec{F}$  is removed. The work done in this case is given by

$$W = Fs \quad \text{constant } \vec{F} \text{ has the same direction as } \vec{s} \quad (6.2)$$

## Units of Work

In the standard *SI* system of units, work has the units of **Newtons** times **Meters**, so one could express the units of work as N-m. There is a special name for the “Newton-meter”, and this name is the **Joule** (J) after the English physicist who did early research into forces and energy.

In the “cgs” system of units (centimeter-gram-second), work has the units of **dynes** times **cm**, and the dyne-cm is given the special name **erg**. There are  $10^7$  ergs in one joule of energy.

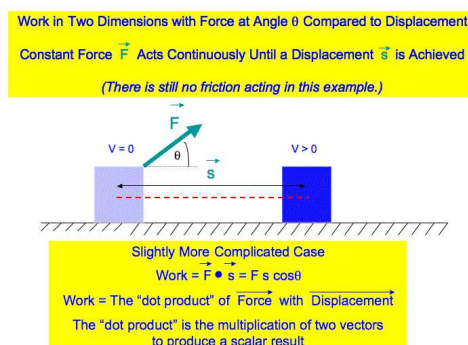
There is also the kilowatt-hour (what you get charged for in your electric bill) which is energy NOT power.  $\text{A kW-hr} = 3.6 \times 10^6 \text{ J}$ .

Finally, there is the atomic, nuclear, and high energy physics units called the **electron-volt** abbreviated as **eV**, the million-electron-volt unit **MeV**, and lastly the giga-electron-volt unit **GeV** =  $10^9 \text{ eV}$ . An **eV** =  $1.6 \times 10^{-19} \text{ J}$ .



## Work in Two (and even Three) Dimensions

The next simplest case of Work is for two dimensions when there is a constant Force but the displacement vector is not in the same direction as the force vector.



Consider again an object resting on a frictionless table. In this case the applied force is not in the horizontal direction, but at some angle  $\theta$  to the horizontal. Also, the vertical component of the applied force ( $= F \sin \theta$ ) is not sufficient to raise the object off the table. Since there is no friction the horizontal component of the force ( $= F \cos \theta$ ) causes the object to accelerate along the horizontal. *In this case the work is done only by the horizontal component of the force.* It is that component of the force which moves the object in the same direction as the component itself.

$$W = (F \cos \theta)s = F s \cos \theta \quad \text{constant } \vec{F} \text{ separated by angle } \theta \text{ from } \vec{s} \quad (6.1)$$

$$W = \vec{F} \cdot \vec{s} \quad \text{Work is the "dot product" of the vectors } \vec{F} \text{ and } \vec{s} \quad (6.3)$$

## The Dot Product of two vectors and the Integral formula for Work

The *dot product* is a means of multiplying two vectors quantities in order to produce a scalar quantity. The dot product is equal to the product of the magnitudes of the two vectors times the cosine of the angle between the two vectors. For a constant  $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ , and  $\vec{s} = s_x \hat{i} + s_y \hat{j} + s_z \hat{k}$  then

$$W = \vec{F} \cdot \vec{s} = F s \cos \theta = F_x s_x + F_y s_y + F_z s_z$$

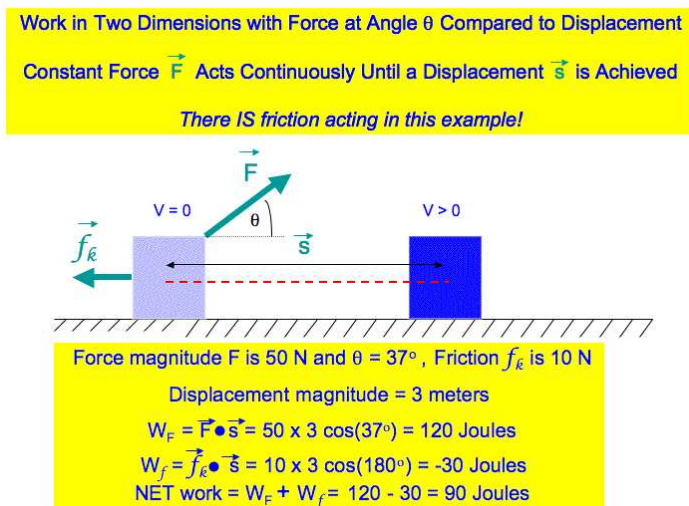
If instead of a *finite* displacement vector  $\vec{s}$  one has a series of *infinitesimal* displacement vectors  $d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$ , then there is an "infinitesimal work"

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz$$

$$\text{Integration} \quad \Rightarrow \quad W = \int_1^2 dW = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$$

### Example of Work in Two Dimensions

A box is dragged across a rough floor by a constant force of magnitude 50 N. The force makes an angle of  $37^\circ$  with the horizontal. A frictional force of 10 N retards the motion. The box is moved a total of 3 m along the floor. Calculate the work done by all the forces acting on the box.



First indicate in the “free-body” diagram all the forces acting on the box. Then indicate the displacement vector. You should realize right away that the all forces acting in the vertical direction do no work. The only work is done by those forces which act in, or have a component in, the horizontal direction.

Work done by the applied 50 N force

$$W_F = (F \cos \theta)s = (50 \text{ N} \cos 37^\circ)(3 \text{ m}) = 120 \text{ N}\cdot\text{m} = 120 \text{ J}$$

Work done by the frictional 10 N force, which is kinetic friction, is given by

$$W_f = (f_k \cos 180^\circ)(3 \text{ m}) = (-f_k)(3 \text{ m}) = (-10 \text{ N})(3 \text{ m}) = -30 \text{ J}$$

Note the negative sign for the work done “by” the force of friction. The negative sign means that work is done “on” (or against) the friction force rather than by the frictional force. *Forces of friction never produce positive work since there are always opposite to the direction of motion.*

The *net work* in this case is the sum of the work of the 50 N force and the work done against friction

$$W_{\text{net}} = W_F + W_f = 120 - 30 = 90 \text{ J}$$

## Work and Kinetic Energy

Take the same example as before, but now consider that the box has a mass of 5 kg. We define the **kinetic energy** of the box to be the product

$$K = \frac{1}{2}mv^2$$

where  $v$  is the speed of the box after it has been moved 3 m. How can we compute  $v$ , knowing that the box has moved from rest?

The speed  $v$  after the box has moved 3 m can be computed once the acceleration is known. We would use the kinematics equation

$$v^2(x) = v_0^2 + 2a(x - x_0) \implies v^2(x = 3) = 2a(3)$$

The box starts from rest ( $v_0 = 0$ ) and the initial position can be taken as the origin ( $x_0 = 0$ ). How can we compute the acceleration  $a$ ?

The acceleration  $a$  can be computed from Newton's second law if you know the accelerating force and the mass. In this case the accelerating force is along the  $x$  direction and is given by the difference of the applied horizontal force and the friction force

$$F_x = F \cos \theta - f_k = 50 \cos 37^\circ - 10 = 30 \text{ N}$$

$$F_x = ma \quad \text{Newton's second law}$$

$$\implies a = \frac{F_x}{m} = \frac{30 \text{ N}}{5 \text{ kg}} = 6 \text{ m/s}^2$$

**Evaluate the Kinetic Energy** The speed may be calculated now

$$v(x = 3) = \sqrt{2a(3)} = \sqrt{2 \cdot 6 \cdot 3} = 6.0 \text{ m/s}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}5(6)^2 = 90 \text{ J}$$

The kinetic energy of the moving box equals the work done by the forces producing the motion  $\implies$  *Work-Kinetic Energy Theorem*

**KINETIC ENERGY = WORK DONE BY A FORCE**

### Work Done by a Spring

A spring is an example of a device which will generate a linearly increasing force as a function of how much the spring is stretched or how much it is compressed. The force generated by a spring when it is stretched or compressed is given by *Hooke's Law*

$$F_{spring} = -kx$$

$x \equiv$  displacement of the spring from its unstretched position

$k \equiv$  force constant of the spring

Suppose the spring has been compressed from an initial position  $x_i$  to a final position  $x_f$  where  $x_f < x_i$ . The spring opposes this compression and the work done ON the spring is as follows:

$$W = \int_{x_i}^{x_f} F_{spring}(x)dx = \int_{x_i}^{x_f} (-kx)dx = -k \int_{x_i}^{x_f} (x)dx$$

$$W = -\frac{1}{2}kx^2 \Big|_{x=x_f}^{x=x_i} = -\frac{1}{2}k(x_f^2 - x_i^2)$$

**Worked Example** A block is lying on a smooth horizontal surface and is connected to a spring with a force constant  $k$  equal to 80 N/m. The spring has been compressed 3.0 cm from its equilibrium (= unstretched) position. How much work does the spring do on the block when the block moves from  $x_i = -3.0$  to  $x_f = 0.0$  cm?

$$W = -\frac{1}{2}k(x_f^2 - x_i^2) = -\frac{1}{2}(80 \text{ N/m})(0 - .030 \text{ m})^2 = 3.6 \times 10^{-2} \text{ J}$$

Note that the result is positive, meaning that Work is done BY the spring in this case.

## Work, Energy, and Power

The last physical quantity introduced in chapter 6 is **Power**. Power is defined as the rate at which work is done. The more work done in a given amount of time, then the more power that is being produced.

Power can either be *average power* or *instantaneous power*

$$\overline{P} = \frac{\Delta W}{\Delta t}$$

$$P(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

The units of power in the SI system is Joules/second which is defined as a watt (W) after James Watt the inventor of the workable steam engine. There is also the kilowatt which is 1,000 watts. Finally electric “power” bills are usually expressed in terms of “kilowatt–hours”. By the definition of the power unit, you can see that a “kilowatt–hour” is not a unit of power, but rather a unit of energy. What you buy from the electric company is total energy consumption as opposed to rate of energy consumption. It is not how fast you use the electricity at any one time, but rather what is the total amount of electricity which you use

**Worked Example** A weightlifter lifts 250 kg through a vertical distance of 2 m in a time of 1.5 seconds. What is his average power output?

First compute the work done,  $\Delta W$ , and then divide by the time to do that work,  $\Delta t$ . Here the weightlifter exerts a force equal to the weight of 250 kg, and that force is used through a distance of 2 meters.

$$W = Fd = Mgd = 250 \cdot 9.8 \cdot 2 = 4,900 \text{ J}$$

The average power is then

$$\overline{P} = \frac{\Delta W}{\Delta t} = \frac{4,900 \text{ J}}{1.5 \text{ s}} = 3,267 \text{ W} \quad (\text{greater than 4.3 horsepower})$$

The “horsepower unit” is a British system unit which is equal to 746 Watts. In principle it is the power output of a “standard horse” whatever that is.