REVIEW: (Chapter 6) Work and Kinetic Energy

The concept of **WORK** has a very precise definition in physics. **Work** is a physical quantity produced when a **Force** moves an object through a **Displacement** in the same direction as the force. Work is a *scalar* quantity.

In physics, in order to get the **Work** done, there must be a force and there must be motion caused by that force. If the force $\vec{\mathbf{F}}$ is constant, and the displacement $\vec{\mathbf{s}}$ is exactly in the same direction as the direction of the force, then the magnitude of the work is easy to calculate:

$$W = Fs \quad \text{constant force } \vec{\mathbf{F}} \text{ has the same direction as displacement } \vec{\mathbf{s}} \qquad 6.1$$

Work if the force is not in the same direction as the displacement

If the constant force $\vec{\mathbf{F}}$ is not in the same direction as the displacement vector $\vec{\mathbf{s}}$, then the magnitude of the work done is given by

$$W = Fs\cos\theta \quad \text{constant } \vec{\mathbf{F}} \text{ separated by angle } \theta \text{ from } \vec{\mathbf{s}} \qquad (6.2)$$

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} \quad \text{Work is the ``dot product'' of the vectors } \mathbf{F} \text{ and } \mathbf{s} \qquad (6.3)$$

Kinetic Energy The Energy of Moving Objects

Consider the following logical progression: When a force acts on an object of mass m for a specific period of time causing the object to move, then work W is done by the force. If the object has a force acting on it, then it will accelerate. If the object was initially at rest, then after the force stops acting the object will have a speed v. We can define the **Kinetic Energy** K of the object to be

$$K = \frac{1}{2}mv^2 \quad \text{(VERY IMPORTANT EQUATION!)} \qquad (6.5)$$

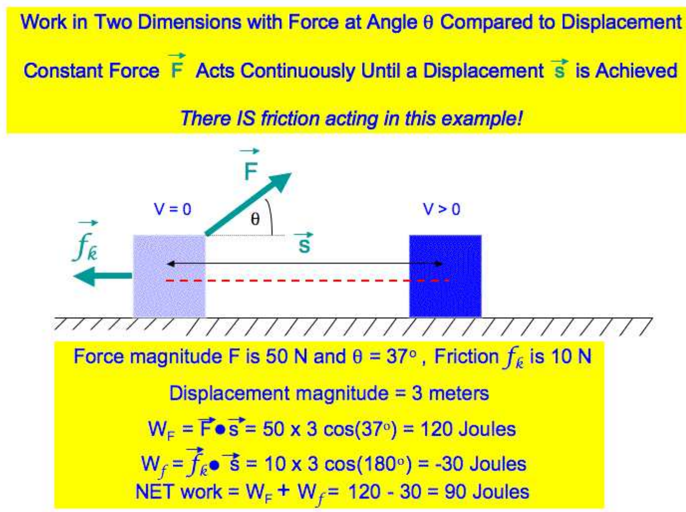
The kinetic energy of an object moving with speed v is equal to the work done by the force which made the object move from rest to the speed v.

In physics **Power** is defined as the **rate** at which work is done by a force. For a constant (time-independent) force $\vec{\mathbf{F}}$, one has the following *instantaneous* rate of work for a changing displacement vector $\vec{\mathbf{s}}$

$$P(t) = \frac{dW}{dt} = \frac{d}{dt}(\vec{\mathbf{F}} \cdot \vec{\mathbf{s}}) = \vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{s}}}{dt} = \vec{\mathbf{F}} \cdot \mathbf{v}(\vec{\mathbf{t}}) \qquad (6.19)$$

REVIEW: Example of Work in Two Dimensions

A box is dragged across a rough floor by a constant force of magnitude 50 N. The force makes an angle of $37^{\rm O}$ with the horizontal. A frictional force of 10 N retards the motion. The box is moved a total of 3 m along the floor. Calculate the work done by all the forces acting on the box.



First indicate in the "free–body" diagram all the forces acting on the box. Then indicate the displacement vector. You should realize right away that the all forces acting in the vertical direction do no work. The only work is done by those forces which act in, or have a component in, the horizontal direction.

Work done by the applied 50 N force

$$W_F = (F\cos\theta)s = (50 \text{ N} \cos 37^{\rm O})(3 \text{ m}) = 120 \text{ N-m} = 120 \text{ J}$$

Work done by the frictional 10 N force, which is kinetic friction, is given by

$$W_f = (f_k \cos 180^{\rm O})(3 \text{ m}) = (-f_k)(3 \text{ m}) = (-10 \text{ N})(3 \text{ m}) = -30 \text{ J}$$

Note the negative sign for the work done "by" the force of friction. The negative sign means that work is done "on" (or against) the friction force rather than by the frictional force. *Forces of friction never produce positive work since there are always opposite to the direction of motion.*

The *net work* in this case is the sum of the work of the 50 N force and the work done against friction

$$W_{net} = W_F + W_f = 120 - 30 = 90 \text{ J}$$

Work and Kinetic Energy

Take the same example as before, but now consider that the box has a mass of 5 kg. We define the **kinetic energy** of the box to be the product

$$K = \frac{1}{2}mv^2$$

where v is the speed of the box after it has been moved 3 m. How can we compute v, knowing that the box has moved from rest?

The speed v after the box has moved 3 m can be computed once the acceleration is known. We would use the kinematics equation

$$v^2(x) = v_0^2 + 2a(x - x_0) \Longrightarrow v^2(x = 3) = 2a(3)$$

The box starts from rest ($v_0 = 0$) and the initial position can be taken as the origin ($x_0 = 0$). How can we compute the acceleration a?

The acceleration a can be computed from Newton's second law if you know the accelerating force and the mass. In this case the accelerating force is along the x direction and is given by the difference of the applied horizontal force and the friction force

$$F_x = F\cos\theta - f_k = 50\cos 37^{\text{O}} - 10 = 30 \text{ N}$$

$$F_x = ma \quad \text{Newton's second law}$$

$$\Longrightarrow a = \frac{F_x}{m} = \frac{30 \text{ N}}{5 \text{ kg}} = 6 \text{ m/s}^2$$

Evaluate the Kinetic Energy The speed may be calculated now

$$v(x = 3) = \sqrt{2a(3)} = \sqrt{2 \cdot 6 \cdot 3} = 6.0 \text{ m/s}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}5(6)^2 = 90 \text{ J}$$

The kinetic energy of the moving box equals the work done by the forces producing the motion $\Longrightarrow$ *Work-Kinetic Energy Theorem*

KINETIC ENERGY = WORK DONE BY A FORCE

Work Done by a Spring

A spring is an example of a device which will generate a linearly increasing force as a function of how much the spring is stretched or how much it is compressed. The force generated by a spring when it is stretched or compressed if given by *Hooke's Law*

$$F_{spring} = -kx$$

$$x \equiv \text{displacement of the spring from its unstretched position}$$
$$k \equiv \text{force constant of the spring}$$

Suppose the spring has been compressed from an initial position x_i to a final position x_f where $x_f < x_i$. The spring opposes this compression and the work done ON the spring is as follows:

$$W = \int_{x_i}^{x_f} F_{spring}(x)dx = \int_{x_i}^{x_f} (-kx)dx = -k\int_{x_i}^{x_f} (x)dx$$

$$W = -\frac{1}{2}kx^2\Big]_{x=x_f}^{x=x_i} = -\frac{1}{2}k(x_f^2 - x_i^2)$$

Worked Example A block is lying on a smooth horizontal surface and is connected to a spring with a force constant k equal to 80 N/m. The spring has been compressed 3.0 cm from its equilibrium (= unstretched) position. How much work does the spring do on the block when the block moves from $x_i = -3.0$ to $x_f = 0.0$ cm?

$$W = -\frac{1}{2}k(x_f^2 - x_i^2) = -\frac{1}{2}(80 \text{ N/m})(0 - .030 \text{ m})^2 = 3.6 \text{ x } 10^{-2} \text{ J}$$

Note that the result is positive, meaning that Work is done BY the spring in this case.

Work, Energy, and Power

The last physical quantity introduced in chapter 6 is **Power**. Power is defined as the rate at which work is done. The more work done in a given amount of time, then the more power that is being produced.

Power can either be *average power* or *instantaneous power*

$$\overline{P} = \frac{\Delta W}{\Delta t}$$

$$P(t) = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

The units of power in the SI system is Joules/second which is defined as a watt (W) after James Watt the inventor of the workable steam engine. There is also the kilowatt which is 1,000 watts. Finally electric "power" bills are usually expressed in terms of "kilowatt–hours". By the definition of the power unit, you can see that a "kilowatt–hour" is not a unit of power, but rather a unit of energy. What you buy from the electric company is total energy consumption as opposed to rate of energy consumption. It is not how fast you use the electricity at any one time, but rather what is the total amount of electricity which you use

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Worked Example A weightlifter lifts 250 kg through a vertical distance of 2 m in a time of 1.5 seconds. What is his average power output?

First compute the work done, ΔW, and then divide by the time to do that work, Δt. Here the weightlifter exerts a force equal to the weight of 250 kg, and that force is used through a distance of 2 meters.

$$W = Fd = Mgd = 250 \cdot 9.8 \cdot 2 = 4,900 \text{ J}$$

The average power is then

$$\overline{P} = \frac{\Delta W}{\Delta t} = \frac{4,900 \text{ J}}{1.5s} = 3,267 \text{ W} \quad \text{(greater than 4.3 horsepower)}$$

The "horsepower unit" is a British system unit which is equal to 746 Watts. In principle it is the power output of a "standard horse" whatever that is.

CHAPTER 7: Potential Energy and Conservation of Energy

The most important principle in all of Physics is the Conservation of Energy.

> *Energy can neither be created or destroyed but only changed from one form into another.*

In Mechanics the two forms of Energy are **Kinetic Energy** and **Potential Energy**. The total Mechanical Energy of a system is equal to the Kinetic Energy plus the Potential Energy.

$$\mathbf{E}_{\text{total}} \equiv \mathbf{K} + \mathbf{U}$$

The Conservation of Mechanical Energy

> *In a system where there are no frictional forces acting, the total Mechanical Energy is constant.*

When there are no frictional forces acting, we say that there are only conservative forces acting. Conservative forces include the force of gravity and the spring (elastic) force.

Kinetic Energy

The **Kinetic Energy** is always defined the same way for any object. If you have an object which has a mass m, and that object is moving with a speed v, then the kinetic energy is always $K = \frac{1}{2}mv^2$

Potential Energy

The two principle forms of **Potential Energy** which we deal with in this chapter are the gravitational potential energy and the elastic potential energy of a spring.

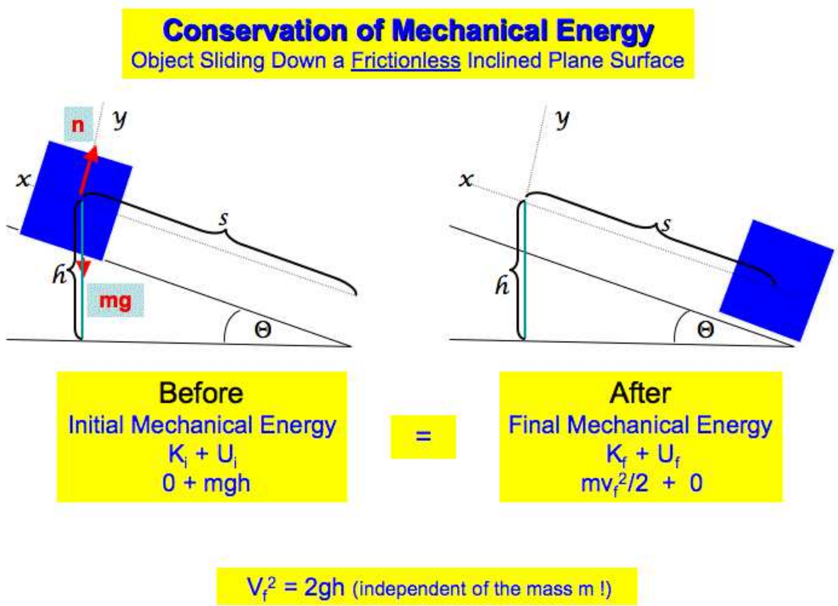
$$U^{gravity} = mgh \qquad (h \text{ is the height above some surface})$$

$$U^{spring} = \frac{1}{2}kx^2 \qquad (x \text{ is the compressed or stretched length})$$

Using the Conservation of Energy to Solve Problems

A mass initially as rest slides down a frictionless inclined plane of height h and inclination angle Θ. What is the speed of the mass when it reaches the bottom of the inclined plane ?

Solution by Conservation of Energy



$$\Longrightarrow v_f = \sqrt{2gh} \quad \text{(independent of the mass !)}$$

Solution by Equations of Motion and Newton's Second Law

Calculate the acceleration a_x down the plane. Then use the third kinematics equation ($v_f^2(x) = v_0^2 + 2a_x(x - x_0)$).

Newton's second law gives the acceleration as the force divided by the mass:

$$a_x = \frac{F_x}{m} = \frac{W \sin\Theta}{m} = \frac{mg\sin\Theta}{m} = g\sin\Theta$$

The total distance traveled is the length of the plane

$$x - x_0 = s = \frac{h}{\sin\Theta}$$

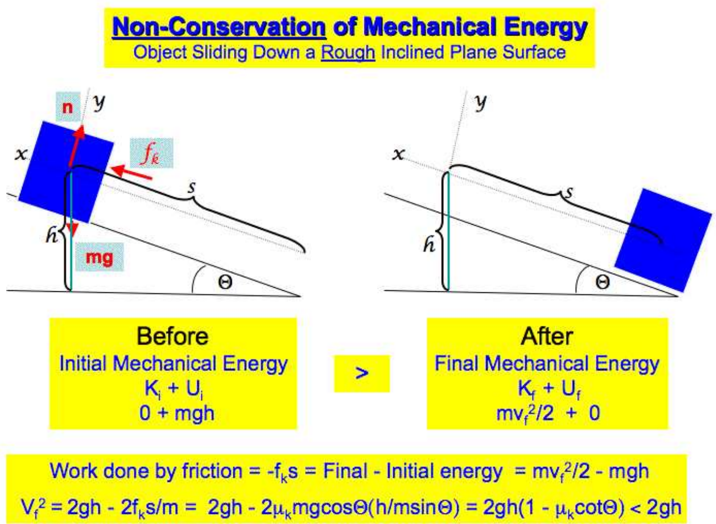
Now substitute into the third kinematics equation

$$v_f^2 = v_0^2 + 2a_x s = 2g\sin\Theta(\frac{h}{\sin\Theta}) = 2gh$$

$$\Longrightarrow v_f = \sqrt{2gh} \quad \text{(same answer but more complicated to derive)}$$

Using the Conservation of Energy with Frictional Forces Present

If there are frictional forces present, then the work done against the frictional (non–conservative) forces is equal to the change (decrease) in the total Mechanical Energy. The total Mechanical Energy is *not* constant when frictional forces are present. The Mechanical Energy will *decrease* because of the work done against the frictional forces.



$$W_{friction} = (K_f + U_f) - (K_i + U_i)$$

Worked Example A 3 kg block slides down a rough incline 1 m in length. The block starts from rest at the top of the inclined plane, and experiences a constant force of friction of magnitude 5 N. The angle of the incline is 30^{O}. Using conservation of energy, determine the speed of the block when it reaches the bottom of the plane.

$$W_{friction} = (K_f + U_f) - (K_i + U_i) \Longrightarrow -fs = (\frac{1}{2}mv_f^2 + 0) - (0 + mgh)$$

The force of friction f is given as 5 N, the length s over which it acts is given as 1.0 m, and the initial height of the block h may be found from simple trigonometry to be 0.5 m.

$$-5 \cdot 1 = \frac{1}{2}3v_f^2 - 3 \cdot 9.8 \cdot 0.5$$

$$v_f^2 = 6.47 \text{ m}^2/\text{s}^2 \Longrightarrow v_f = 2.54 \text{ m/s}$$

In this example, the total Mechanical Energy was **not** conserved because of the non–conservative frictional forces. The decrease in the Mechanical Energy went into doing work against friction, and that work actually would show up as heat.

Using Conservation of Mechanical Energy in Spring Problems

The principle of conservation of Mechanical Energy can also be applied to systems involving springs. First take a simple case of a mass traveling in a horizontal direction at constant speed. The mass strikes a spring and the spring begins to compress slowing down the mass. Eventually the mass stops and the spring is at its maximum compression. At this point the mass has zero kinetic energy and the spring has a maximum of potential energy. Of course, the spring will rebound and the mass will finally be accelerated to the same speed but opposite in direction. The mass has the same kinetic energy as before, and the spring returns to zero potential energy.

Spring Potential Energy
If a spring is compressed (or stretched) a distance x from its normal length, then the spring acquires a potential energy $U^{spring}(x)$:

$$U^{spring}(x) = \frac{1}{2}kx^2 \quad (k = \text{ force constant of the spring})$$

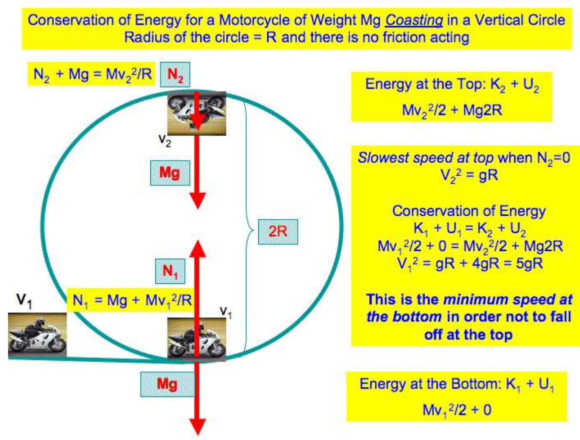
Worked Example A mass of 0.80 kg is given an initial velocity $v_i = 1.2$ m/s to the right, and then collides with a spring of force constant $k = 50$ N/m. Calculate the maximum compression of the spring.
Solution by Conservation of Energy

$$\text{Initial Mechanical Energy} = \text{Final Mechanical Energy}$$
$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_i^2 + 0 = 0 + \frac{1}{2}kx^2$$

$$\Longrightarrow x = v_i\sqrt{\frac{m}{k}} = 1.2\sqrt{\frac{0.8}{50}} = 0.152 \text{ m}$$

Conservation of Energy and the Loop–the–Loop



The loop–the–loop consists of a curved track whose initial point is located a distance h above ground level. The curve goes into a vertical circle of radius R with its bottom most point at ground level. The track and the vertical circle are assumed to be frictionless. Why does the particle sometimes leave the vertical circle before reaching the top most point ? Obtain an expression for the minimum value of h such that the particle will not fall off the vertical circle portion of the track.

We solve this problem in two steps. First we consider what is the minimum speed v_1 necessary at the bottom of the track in order to coast to the top of the track with a speed v_2 which is fast enough. As the bike coasts to the top of the track, it is exchanging kinetic energy for potential energy. We have see previously (see the figure above) that at the top of the track the minimum speed is the speed at which the weight force is dedicated to providing all the centripetal force. Any slower speed will cause the bike to fall off the track. With this condition, the minimum speed at the top of the track is

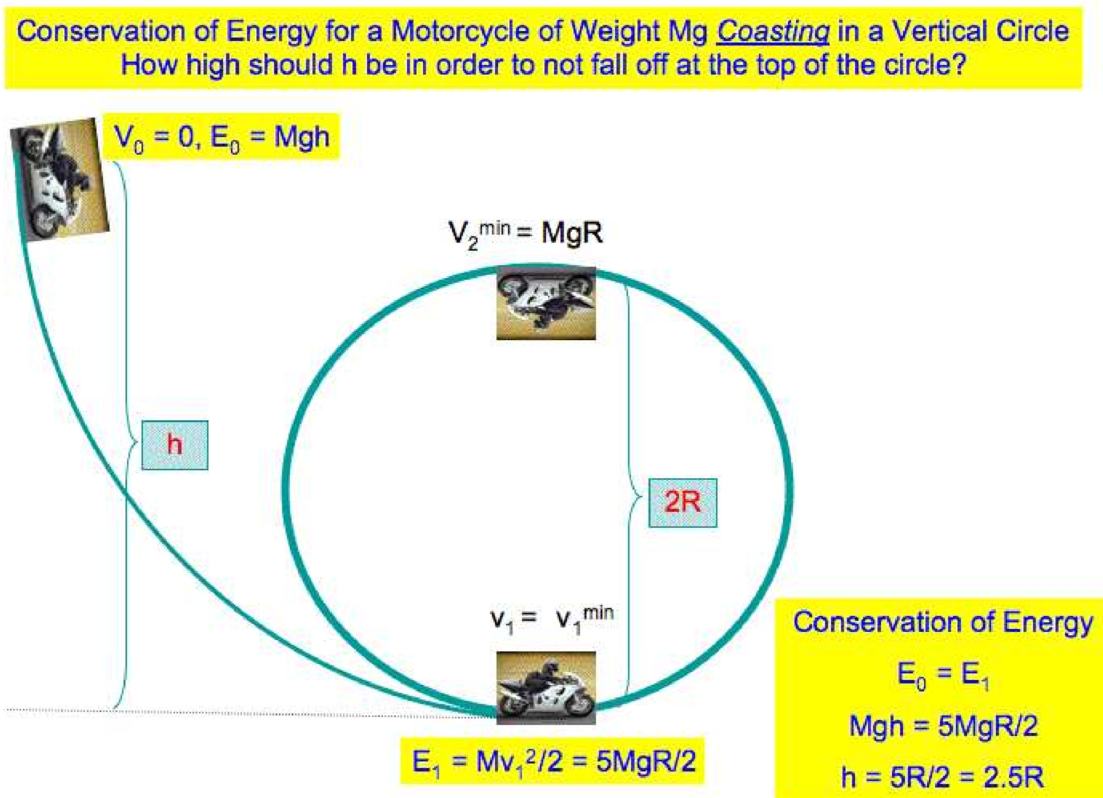
$$v_2^{min} = \sqrt{gR}$$

Knowing that this is the minimum speed, we can use conservation of mechanical energy in order to derive the minimum speed at the bottom of the track. Essentially the bike will gain $2MgR$ worth of potential energy in going to the top of the track. From the figure above we see that the minimum speed

$$v_1^{min} = \sqrt{5gR}$$

The next step will be to find out how high a bike must be to have a potential energy equal to the kinetic energy at this speed v_1^{min}.

Conservation of Energy and the Loop–the–Loop



In the above figure we show the bike at an initial vertical height h above the bottom of the circular track. The bike starts at 0 speed from this height. We can use conservation of energy to find the speed at the bottom.

$$Mgh = \frac{1}{2}Mv_1^2$$

$$v_1 = \sqrt{2gh}$$

If we have $v_1 = \sqrt{5gR}$, then

$$5gR = 2gh$$

$$h = 2.5R$$

This is the minimum height to drop a mass in order that the mass do a loop-the-loop where the loop has a radius R.

If you can understand this problem, you know a lot.