CHAPTER 13: Wave Motion

A **wave** is the propagation of energy (motion) through a medium. When a wave propagates, the medium is disturbed from its equilibrium position for a short period of time and then returns to its normal position. Think of the "wave" which travels around the fans in a football stadium. The medium here is the football fans, and the motion is a pulse movement standing up and then sitting down.

A **transverse wave** propagates in a direction perpendicular to the motion of the medium (again think of the football wave). Water waves are a good example of transverse waves. A **longitudinal wave** has its motion in the direction of the displacement. Sound waves are a good example of longitudinal wave motion.

No matter what the wave, *there is no net displacement of the particles in the medium once the wave has passed.*

A wave is characterized by a displacement y which occurs at a given position x and at a given time t. Therefore, in order to describe mathematically the equation of a wave, one must write y as a function of x and t:

$$y(x,t) = f(kx - \omega t) \quad \text{(wave traveling in the } +x \text{ direction)} \qquad (13.1)$$

$$y(x,t) = f(kx + \omega t) \quad \text{(wave traveling in the } -x \text{ direction)} \qquad (13.2)$$

Two or more waves may be traveling through a single medium. When these waves pass through one another, the exhibit the phenomenon of **interference** which is a consequence of the **superposition principle**. When there are two waves at the same position x at the same time t, the net displacement is the algebraic sum of the displacements of the two individual waves.

For the case of a stretched (guitar) string or cable under a tension F and having a mass M and length L, the speed of a wave pulse is

$$v = \sqrt{\frac{F}{\mu}} \text{ where } \mu \equiv \frac{M}{L} \quad ; \quad v = f\lambda = \frac{\lambda}{T} \text{ (harmonic waves)} \qquad (13.20, 13.11)$$

The Two Types of Waves

Transverse Waves

The most familiar type of wave motion is that of waves on a beach. This motion should give you a good idea of the wave phenomenon. It is the transmission of energy, manifested by the up and down motion of the water, through a medium. Think of a seagull or a duck floating on the water. Before a wave hits, the bird is motionless. Then the bird is successively raised up and lowered by the moving water, and finally the bird goes back to its original height. The same is true of the water itself. Except when the wave is passing through, the molecules of the water are undisturbed from their positions. They occupy the same positions after the wave as before.

This type of water wave is a **transverse wave**. The energy contained in the wave pulse causes the medium to move up and down which is perpendicular to the direction in which the wave pulse is propagating.

Wave motion is an important subject of study because it is the basis for all our electronic communication. Light itself is a wave phenomenon, and the heat from the sun reaches us by *infra–red rays*, which is all part of the same theory of electric and magnetic waves.

Longitudinal Waves

The second type of wave motion is **longitudinal waves**. In this type of wave propagation, the energy of the wave causes the medium to move back and forth in a direction parallel to the direction of propagation of the wave. Sound waves are the most important examples of **longitudinal wave** motion. Another example would be the effect of a gust of wind blowing through a field of wheat. One can see the stalks of wheat “rippling” in the field, moving back and forth, as the gust moves through. Thus the song phrase “amber waves of grain”.

The Equation of Motion of a Traveling Wave in One Dimension

Fundamentally a wave travels. We consider first the simplest case of a *sinusoidal* wave traveling in only one dimension, say x. The displacement that the wave causes we say is in the y direction. The wave is characterized by an amplitude y_m, a wavelength λ and a frequency f, as we saw in the previous lecture.

$$\omega = 2\pi f \quad ; \quad T = \frac{1}{f} \tag{13.8}$$

The displacement y must be a function of both the position x and the time t. If the shape of the wave does not change as it moves along, then we can write a special form of this dependence for a *sinusoidal wave*:

$$y(x,t) = y_m \sin(kx - \omega t) \tag{13.9}$$

Fig. 13.8 gives "snapshots" of a traveling wave on the x and the t axes. The parameter k is related to the wavelength $k \equiv 2\pi/\lambda$(Eq. 13.7).

The speed of a traveling wave

The speed of a traveling wave is given by the important formula:

$$v = \lambda f \tag{13.11}$$

This can be obtained by looking at consecutive snapshots of the traveling wave (see Fig. 13.8 separated by a time Δt and then computing the distance Δx traveled in that time. The speed will be the ratio of $\Delta x/\Delta t$.

Using the Wave Equation

Consider the sinusoidal wave given by the formula

$$y(x,t) = 0.00327 \sin(72.1x - 2.72t)$$

where the three numerical constants are in meters, rad/meter, and rad/s. What is the *amplitude*, the *wavelength*, the *frequency*, and the *speed* of this wave?

To solve this, all one has to do is compare with the basic sinusoidal wave equation:

$$y(x,t) = y_m \sin(kx - \omega t) \tag{13.9}$$

Then it is simply a matter of comparing the components of this equation with those in the example:

$$\begin{aligned}
\textbf{coefficient of sine function} \quad &= \quad y_m = 0.00327 \text{ meters} \\
\textbf{coeffiecient of x} \quad &= \quad k = 72.1 \text{ rad/m} \quad \Longrightarrow \lambda = \frac{2\pi}{k} = 0.0871 \text{ meters} \\
\textbf{coefficient of t} \quad &= \quad \omega = 2.72 \text{ rad/s} \quad \Longrightarrow f = \frac{\omega}{2\pi} = 0.433 \text{ Hz}
\end{aligned}$$

The wave speed can be computed from $v = \lambda f$

$$v = \lambda f = 0.0871 \cdot 0.433 = 0.0377 \text{ m/s}$$

The Velocity of Waves on a String Under Tension

This being "Music City" we all know about guitars and guitar strings. Guitar strings, which can be metallic or non–metallic, are of varying lengths according to the pitch (frequency) of the sound. The guitar string is tuned to the correct sound by changing slightly the tension using a turn screw in the neck of the guitar at the end of the string. The reason that this works is that the tension of the string determines the velocity of the wave in which is generated when the guitar string is plucked.

$$v = \sqrt{\frac{F}{\mu}} \tag{13.20}$$

where μ is the mass per unit length of the string $\mu = M/L$. By making the tension greater, then the velocity increases. In turn, an increase in velocity leads to a higher frequency f of the sound.

$$v = f\lambda \quad \Longrightarrow \quad f = \frac{v}{\lambda} \tag{13.13}$$

where the **wavelength** of the wave motion in the guitar string is fixed by the length of the string, which we will discuss later.

Worked Example of a Wave Traveling on a String

A uniform string has a mass of 0.3 kg and a length of 6 m. Tension is maintained on the string by suspending a 2 kg mass from one end. Find the speed of a sinusoidal wave in the string:

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{F}{M/L}} = \sqrt{\frac{(2.0 \cdot 9.8)}{0.3/6}} = 19.8 \text{ m/s}$$

Superposition and Interference of Waves

A very interesting facet of wave theory is the **superposition principle**

If two or more traveling waves are moving through a medium, the resultant wave function at any point is the algebraic sum of the wave functions of the individual waves. What this means is that two waves traveling in a string act independently of one another. Two waves can even pass through one another without disturbing their individual shapes. The addition (algebraic sum) of the two waves is called **interference**. For example, if the peak of one wave meets the minimum of a second wave of equal amplitude, then there will be no net displacement of the medium. This is called **destructive interference**. On the other hand if the peak of one wave meets the peak of a second wave, or if the minima of the two waves coincide, then the two waves reinforce each other and the net displacement is doubled. This is called **constructive interference**. When two waves traveling in the same direction, with the same amplitude y_m, the same angular frequency ω, and the same wavelength $\lambda = 2\pi/k$, but separated by a phase difference ϕ meet at the same place, they will add algebraically (superposition)

Reflection and Transmission of Waves

Another interesting phenomenon about wave motion is what happens when a wave pulse traveling along a string hits a solid wall to which the string is firmly attached. What happens is that the wave is *reflected and inverted.* On the other hand, if a wave is traveling along a string and gets to the end of the string which is not held tight but is free to move, then the wave is *reflected but not inverted.*

A third possibility is the intermediate case. Suppose that there are two strings of different densities which are tied together. A wave pulse is traveling along the first (lighter) string. In that case there will be an inverted reflected inverted wave along the first string, and a non–inverted transmitted wave along the second string. On the other hand, if the wave pulse is first traveling along the heavier string, both the transmitted and the reflected waves will be non–inverted.

When a wave pulse travels from medium A to medium B, and medium B is denser than medium A ($\Longrightarrow v_A > v_B$), then the reflected wave is inverted. Conversely, if medium A is more dense than medium B ($\Longrightarrow v_B > v_A$), then the reflected wave is non–inverted. In either case, the transmitted wave is non–inverted. What about the amplitudes of the reflected and transmitted waves?

Transverse Velocity, Acceleration, and Energy in Waves

We have defined waves as energy propagating through matter. Now we will see exactly how much energy we have in a wave. First we start with the simplified formula for a harmonic wave:

$$y(x,t) = A\sin(kx - \omega t) \tag{13.9}$$

Now we compute the velocity in the y direction

$$v_y = \frac{dy}{dt}_{x=\text{constant}} = \frac{d}{dt}\Big(A\sin(kx - \omega t)\Big) = -\omega A\cos(kx - \omega t)$$

This formula resembles very much the formula for the velocity in simple harmonic motion, and that's exactly why these are called **harmonic waves**. We remember that the maximum velocity in SHM is given by

$$v_y^{max} = \omega A$$

Next, we take an element length along the x direction, Δx, and look at how much energy is contained in that element. From our study of simple harmonic motion we know that the total energy is given by one–half the mass times the square of the maximum velocity

$$\Delta E = \frac{1}{2}(\Delta m)(v_y^{max})^2 = \frac{1}{2}(\Delta m)(\omega A)^2$$

This is the energy contained in a length Δx which contains an amount Δm of mass.

$$\Delta E = \frac{1}{2}(\Delta m)\omega^2 A^2 = \frac{1}{2}(\mu\Delta x)\omega^2 A^2$$

Lastly we can compute the power in a wave, how much energy is being transmitted per unit time

$$P = \frac{dE}{dt} = \frac{1}{2}\mu\frac{\Delta x}{\Delta t}\omega^2 A^2 = \frac{1}{2}\mu v\omega^2 A^2 \tag{13.23}$$

The power in a wave is linearly proportional to the velocity of the wave, and *quadratically* proportional to the amplitude of the wave. So if you are in the surf and you see a wave coming which is twice as high as the previous waves, prepare to have yourself clobbered with four times as much energy.