REVIEW: Going from **ONE** to **TWO** Dimensions with Kinematics

In Lecture 2, we studied the motion of a particle in just **one dimension**. The concepts of velocity and acceleration were introduced. For the case of constant acceleration, the **kinematic** equations were derived so that at any instant of time, you could know the position, velocity, and acceleration of a particle in terms of the initial position and the initial velocity. Now the same thing will be done in **two dimensions**. It is important that you recall what you have learned in the one dimension case.

Review of one dimension, constant acceleration kinematics

In one dimension, all you need to know is the position and velocity at a given instant of time:

$$x(t) = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$$

$$v_x(t) = v_{x0} + a_x t$$

Note that I have put a sub–script x in these above equations. For strictly one dimensional motion, such a sub–script is superfluous. However, it is useful in extending your knowledge of kinematics to two dimensions.

Extension of kinematics to two dimensions

In two dimensions, say X and Y, you need to know the position and velocity of particle as a function of time in two, **separate** coordinates. The particle, instead of being confined to travel only along a straight horizontal (or vertical) line, is now allowed to move in a plane. The extension of kinematics to two dimensions is very straightforward.

For the X coordinate:

$$x(t) = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$$

$$v_x(t) = v_{x0} + a_x t$$

For the Y coordinate:

$$y(t) = y_0 + v_{y0}t + \frac{1}{2}a_y t^2$$

$$v_y(t) = v_{y0} + a_y t$$

Worked Example of Two Dimensional Motion pages 79–80

A particle moves in the xy plane with an x component of acceleration $a_x = 4\ \mathrm{m/s^2}$. The particle starts from the origin at $t = 0$ with an initial velocity having an x component of 20 m/s, and a y component of -15 m/s. (There is no y component of acceleration $\Longrightarrow a_y = 0$.)

a) What are the x and y components of the velocity vector as a function of time ?

b) What are the velocity and speed of the particle at $t = 5$ s ?

c) What are the x and y components of the position vector as a function of time ?

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a) Velocity kinematics equations:

$$v_x(t) = v_{x0} + a_x t = 20\ \mathrm{m/s} + 4\ \mathrm{m/s}^2 t$$

$$v_y(t) = v_{y0} + a_y t = -15\ \mathrm{m/s}$$

$$\mathbf{v(t)} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}} = (20\ \mathrm{m/s} + 4\ \mathrm{m/s}^2 t)\,\hat{\mathbf{i}} - (15\ \mathrm{m/s})\,\hat{\mathbf{j}}$$

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b) Velocity and speed at $t = 5$ s

Substitute for $t = 5$ s in the above equations for $\mathbf{v(t)}$

$$\mathbf{v(t=5)} = (20 + 4(5))\,\hat{\mathbf{i}} - 15\,\hat{\mathbf{j}} = 40\ \mathrm{m/s}\,\hat{\mathbf{i}} - 15\ \mathrm{m/s}\,\hat{\mathbf{j}}$$

The *speed* is obtained by using, again, the Pythagorean theorem

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(40)^2 + (-15)^2} = 42.7\ \mathrm{m/s}$$

$$\Theta_v = \tan^{-1}\frac{v_y}{v_x} = \frac{-15}{40} = -20.6^{\mathrm{O}}$$

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b) What are the velocity and speed of the particle at $t = 5$ s ?

c) What are the x and y components of the position vector as a function of time ?

c) The separate position functions $x(t)$ and $y(t)$, given that at $t = 0$ the values are $x_0 = 0$ and $y_0 = 0$:

$$x(t) = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \Longrightarrow x(t) = 20(\text{ m/s})t + 2(\text{ m/s}^2)t^2$$

$$y(t) = y_0 + v_{x0}t + \frac{1}{2}a_y t^2 \Longrightarrow y(t) = -15(\text{ m/s})t$$

The general position vector $\mathbf{r}(\mathbf{t})$ is then given by

$$\mathbf{r}(\mathbf{t}) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} = (20(\text{ m/s})t + 2(\text{ m/s}^2)t^2)\,\hat{\mathbf{i}} + (-15(\text{ m/s})t)\,\hat{\mathbf{j}}$$

We can determine the velocity vector $\vec{v}(t)$ by taking the time derivative of the position vector:

$$\mathbf{v}(\mathbf{t}) = \frac{d\mathbf{r}(\mathbf{t})}{dt} = (20 + 4t)\,\hat{\mathbf{i}} - 15\,\hat{\mathbf{j}}$$

Special Case of **PROJECTILE** Motion

There is a special, very important case of **two dimensional** motion with constant acceleration. This is the case of projectile motion for which the vertical motion is governed by gravity, *and there is no acceleration in the horizontal direction.* So what can one say about the velocity in the horizontal direction?

Projectile Motion, no horizontal acceleration, $a_x = 0$

$$v_x(t) = v_{x0} + a_x t \Longrightarrow v_x(t) = v_{x0}$$
$$x(t) = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \Longrightarrow x(t) = x_0 + v_{x0}t$$

general kinematic equations $\Longrightarrow$ specific projectile motion equations

In *projectile motion*, the horizontal velocity is constant and remains equal to the initial velocity. It is most important that you realize and remember that fact. By consequence, the distance traveled horizontally increases linearly with the duration of the time traveled.

Projectile Motion, vertical acceleration $= a_y = -\mathbf{g}$

$$v_y(t) = v_{y0} - gt$$
$$y(t) = y_0 + v_{y0}t - \frac{1}{2}gt^2$$

Special Case of **PROJECTILE** Motion: Common time parameter

In the projectile equations we have written separately the x position as a function of time, $x(t)$, and the y position as a function of time, $y(t)$. Now in each equation it is the same *time* that we are using. It is exactly the same tick on the clock or number on the digital watch that is being used. So, we may solved for the time variable from the $x(t)$ equation, and substitute that in the $y(t)$ equation.

$$x(t) = x_0 + v_{x0}t \Longrightarrow t = \frac{x}{v_{x0}}$$

Now substitute this in the $y(t)$ equation.

$$y(t) = y_0 + v_{y0}t - \frac{1}{2}gt^2 = y_0 + v_{y0}(\frac{x}{v_{x0}}) - \frac{1}{2}g(\frac{x}{v_{x0}})^2$$

$$y(x) = y_0 + \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}\frac{g}{v_{x0}^2}x^2$$

The Trajectory Equation

The position Y as a function of the position X is given a special name:
the **trajectory equation**

$$y(x) = y_0 + \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}\frac{g}{v_{x0}^2}x^2$$

The motion of a **projectile**, in terms of the x and the y positions is a **parabola**. Notice, that on the left side, we have switched from writing $y(t)$ to $y(x)$ because on the right hand side we have eliminated the time coordinate t in favor of the position coordinate x.

Prototype Problem for Projectile Motion

Suppose a cannon is fired at ground level with some initial velocity $\vec{v}$. That is, the cannonball exits the cannon with a speed v at some angle θ with respect to the horizontal axis. Describe the motion of the cannonball. Specifically

1) How high h (vertical direction) does the cannonball go ?
2) How far R horizontal direction does the cannonball go?
3) How much time t_1 does it take for the cannonball to reach its maximum height?
4) How much time does it take before falling back to the ground?

How high h does the cannonball go?

First we realize that the cannonball executes a parabolic path (see Fig. 3.7 on page 83 in the text).
At the highest point of the trajectory we know that its vertical velocity component is 0. So we have

$$v_y(t_1) = 0 = v_{y0} - gt_1 \Longrightarrow t_1 = \frac{v_{y0}}{g} = \frac{v_0 \sin\theta_0}{g}$$

where we take t_1 to be the time to reach the maximum height. Note: $v_{y0} = v_0 \sin\theta_0$ where θ_0 is the initial direction of the velocity vector.
Now we can use this time t_1 to substitute into the vertical position equation:

$$y(t_1) = h = y_0 + v_{y0}t_1 - \frac{1}{2}gt_1^2$$

$$h = 0 + v_{y0} \cdot \frac{v_{y0}}{g} - \frac{1}{2}g(\frac{v_{y0}}{g})^2 = \frac{v_{y0}^2}{2g} = \frac{v_0^2 \sin^2\theta_0}{2g}$$

So we have answered parts 3) and 1) above, and we should be able to quickly get the answer for part 4). *What is the answer to part 4)?*

Projectile Equations: Initial Position at Origin

The **trajectory equation** was previously shown to be:

$$y(x) = y_0 + \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}\frac{g}{v_{x0}^2}x^2 \tag{3.13}$$

In terms of the cannonball problem, where we specify v and θ, instead of v_{x0} and v_{y0}, and have $x_0 = 0 = y_0$ this is easily transformed to be:

$$y(x) = (\tan\theta_0)x - \Big(\frac{g}{2v_0^2\cos^2\theta_0}\Big)x^2 \tag{3.14}$$

Horizontal Range

A distance of interest is the **Horizontal Range** which the text symbolizes with the letter R. The horizontal range is the x distance which the projectile travels before returning to the ground level. The solution for R can be obtained by solving trajectory equation for $y(R) = 0$:

$$y(R) = 0 = (\tan\theta_0)R - \Big(\frac{g}{2v_0^2\cos^2\theta_0}\Big)R^2$$

One solution is $R = 0$ (why?), and the other solution is given by

$$R = \frac{v_0^2\sin 2\theta_0}{g} \tag{3.16}$$

Note that the maximum value of the Range occurs at $\theta_0 = 45^{\mathrm{O}}$, and is given by:

$$R_{max}(\theta_0 = 45^{\mathrm{O}}) = \frac{v_0^2}{g}$$

You should also convince yourself that different values θ_0 which are symmetric about $\theta_0 = 45^{\mathrm{O}}$ will give the same value for the Range.

Worked Example: "Shoot–the–Monkey" demonstration In the first edition of this text there was a photograph of what is sometimes called the "Shoot–the–Monkey" demonstration. The basic idea is that a hunter spies a monkey hanging from a tree branch. The hunter knows that when he fires his rifle, the monkey will drop instantaneously from the tree. Where should the hunter aim his rifle: 1) above the monkey, 2) at the monkey, or 3) below the monkey?

How do you quantify the fact that an "intercept" has occurred, in other words the projectile fired from the gun hits the dropping target ?

An intercept will occur if *at the same time* the projectile and the target are at exactly the same coordinates (x, y)

1) Since the target is just dropping, its horizontal position remains the same at all time: $x \equiv x_T$

2) Now calculate how long it takes the projectile to reach the $x = x_T$

$$x(t) = v_{x0}t = (v_0 \cos\theta_0)t$$

$$\Longrightarrow t(intercept) \equiv t_I = \frac{x_T}{v_0 \cos\theta_0}$$

3) Now calculate the vertical position y_P where the projectile is at $t = t_I$

$$y_P(t_I) = v_{y0}t_I - \frac{1}{2}gt_I^2 = (v_0 \sin\theta_0)(\frac{x_T}{v_0 \cos\theta_0}) - \frac{1}{2}g(\frac{x_T}{v_0 \cos\theta_0})^2$$

$$y_P(t_I) = x_T \tan\theta_0 - \frac{1}{2}g\Big(\frac{x_T}{v_0 \cos\theta_0}\Big)^2$$

4) For the dropping target, its initial height $y_0 = x_T \tan\theta_0$, its initial velocity is 0, and so its position at $t = t_I$ is given by:

$$y_T(t_I) = y_0 - \frac{1}{2}gt^2 = x_T \tan\theta_0 - \frac{1}{2}g\Big(\frac{x_T}{v_0 \cos\theta_0}\Big)^2$$

So both the target and the projectile meet at the same (x, y) coordinates simultaneously. Notice that this result is independent of v_0.

Circular Motion and Motion in a Curved Path

We have stated that acceleration is the time rate of change of vector velocity

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\vec{\Delta v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Now there are two ways that one can get a non-zero value for $\vec{\Delta v}$. The first, and most obvious, way is to have a change in the magnitude of $\vec{v}$. This is what we normally think of as acceleration: an increase (decrease deceleration) of speed. However, and this is not so obvious at first glance, we can also get a change in the velocity vector *even if the magnitude v does NOT change.* How is this possible. Simple. Just change the direction of the velocity vector $\vec{v}$. The change in the direction of $\vec{v}$, even if the magnitude v stays constant, produces a $\vec{\Delta v}$.

Motion in a Circle at Constant Speed

The simplest case of changing the direction of the velocity vector without changing the magnitude v is to have motion in a circle of constant radius r at a constant speed v. According to the above discussion, we must then have an acceleration. The magnitude of this acceleration is easily proven (page 87–89) to be:

$$a_c = \frac{v^2}{r} \tag{3.17}$$

Note that we have attached a subscript c to the symbol for this acceleration. The reason to use this subscript is to indicate the *direction* which in this case is *along the radius towards the center of the circle.* This kind of acceleration is called *centripetal acceleration* meaning center-seeking acceleration.

Acceleration Has Two Components

In two dimensional motion acceleration has two *Cartesian* components $\vec{a} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}}$. The term Cartesian means using a rectangular coordinate system. However, in some cases it is more useful to think of the *tangential* and the *radial* components of the acceleration. It is still the same acceleration, but expressed in a different coordinate system.

$$\vec{a} = \vec{a}_t + \vec{a}_c$$

The *tangential component* is in the direction *tangent* to the path. This component of acceleration increases the magnitude of the velocity. The *radial* or *centripetal* component of the acceleration changes the direction of the velocity vector.

Relative Velocity

Different Coordinate Frames

As we saw on the last page, the same vector may be decomposed into its components in different coordinate systems. The idea of coordinate systems is an important one which we have been using so far without too much thought. However, the subject of coordinate systems (also called reference frames) is extremely important in Physics. In fact, it was by a study of how Physics is derived in different reference frames that Albert Einstein came up with his famous Theory of Relativity. We will study Relativity in detail (Chapter 9) right after Spring Break. For now we just show some simple examples.

Questions About Moving Reference Frames

Suppose you are in a train, and the window shades down and the track is very smooth, quiet, *and* straight. Can you tell that you are moving? For that matter, seated in the classroom, can you tell that the Earth is moving around the Sun? Suppose the train goes around a sharp curve, again with the window shades still shut. Can you tell whether this is happening?
Same train and on a straight track, but the window shades are up and you can see that you are moving very fast, say 100 miles/hour. You decide to stand in the aisle and jump straight up as high as you can. Where to you land in the aisle? What does your motion look like to a fellow passenger? What does your motion look like to someone looking through the window?

Velocities and Moving Reference Systems

Suppose that there is one coordinate system O' moving at a constant velocity $\vec{V}_{OO'}$ with respect to another coordinate system O. Now a particle P is observed to be moving in coordinate system O with velocity $\vec{v}_{PO}$. In the O' coordinate system an observer will see velocity $\vec{v}_{PO'}$. These three velocities are related by

$$\vec{v}_{PO'} = \vec{v}_{PO} - \vec{V}_{OO'} \qquad 3.22$$

The above equation is called the *Galilean Velocity Transformation.* It was named after Galileo, was was thought to be universally true up until Einstein found that it was not correct at velocities near the speed of light. However, you can use this velocity transformation equation quite well at normal speeds. In fact, there are interesting relative velocity problems such as flying in a cross-wind, or a boat crossing a river with a current which we can study. You already know how to solve these problems qualitatively, if not quantitatively, by your own experiences.