CHAPTER 2: Reporting and Using Uncertainties

Quoting a result as: Best Estimate ± Uncertainty

In the Archimedes experiment result, we had a table which read

Measurement of Crown Density by Two Experts

Measurement reported	Expert A	Expert B
Best estimate for ρ_{crown}	15	13.9
Estimated range for ρ_{crown}	13.5 – 16.5	13.7 – 14.1

Instead of giving a table, most experimenters would quote the two results as follows:

$$\rho_{\text{A}} = 15.0 \pm 1.5 \text{ gm/cm}^3$$

$$\rho_{\text{B}} = 13.9 \pm 0.2 \text{ gm/cm}^3$$

That is, we quote a best (most probable) value of the quantity and then indicate an error band about this best value with the ± symbol.

You might well ask *What is the quantitative meaning of the uncertainty value?* The answer to that question is derived in Chapter 4 of the text. For now we can simply say that the odds are that if you repeated the measurement many times then about 2/3 of these repeated measurements would fall within the error band about the best value you are quoting.

Use of Significant Figures

When quoting the Best Value and the Uncertainty, you have to be careful to specify the correct number of significant figures for each. If you don't, then to the practiced eye your wrong number of significant figures looks plain silly.

For example, in the above quote for ρ_{A}, it would be silly to write

$$\rho_{\text{A}} = 15.0 \pm 1.5234 \text{ gm/cm}^3$$

Once you have said that the error is 1.5 gm/cm^3, it does not add anything more to have the extra decimal places. By the same token, it would be ridiculous to write the best value as

$$\rho_{\text{A}} = 15.01 \pm 1.5 \text{ gm/cm}^3$$

With an error of ±1.5, it does not make any sense to quote the best value to the hundredths decimal place.

Reporting Best Values and Their Uncertainties

Use of Significant Figures

So the procedure is to quote your error to one or two significant figures, and then let that error quote determine how many significant figures are to be attached to your Best Value quote. For example, if you have an error of ±30 m/s in a speed measurement, then the following would be a correct quote of the best value if your average value was 6047.1 m/s

$$v = 6050 \pm 30 \text{ m/s}$$

If your error was ±3 m/s, then you would quote the result as

$$v = 6047 \pm 3 \text{ m/s}$$

Scientific Notation

Of course, with some measurements you will have to use scientific notation. For example supposed you measured the charge on an electron to be 1.61×10^{-19} Coulombs with an estimated error of 5.0×10^{-21} Coulombs. The most meaningful way to write this result is

$$(1.61 \pm 0.05) \times 10^{-19}\text{Coulomb}$$

whereas

$$1.61 \times 10^{-19} \pm 5.0 \times 10^{-21}\text{Coulomb}$$

is very hard to read.

Comparison of Experiment Values with True Values

Discrepancy Between Measured and Accepted Values

In many undergraduate labs you will be determining physical values whose results are already more accurately known than you could measurement them with normal lab apparatus. In such cases we say that there is an *accepted value*, and we ignore the error in the accepted value. For these experiments we are interested in the *discrepancy* between the accepted value and the value obtained in the lab experiment.

As a trivial example, you might be measuring the speed of sound in air for which the accepted value at standard temperature and pressure is

$$v_{\text{acc}} = 331 \text{ m/s}$$

Your measurement, with quoted error is given as

$$v_{\text{exp}} = 329 \pm 5 \text{ m/s}$$

So there is a *discrepancy* of 2 m/s between the measured value and the accepted value. However, this discrepancy is well within the uncertainty of the measurement. So there is no reason to think that the measurement procedure or the data analysis was done wrong.

On the other hand, it might happen that your measurement with quoted error is given as

$$v_{\text{exp}} = 345 \pm 3 \text{ m/s}$$

In this case the discrepancy of 14 m/s is well outside the uncertainty of the measurement. This probably indicates that the measurement was done wrong, or there is a mistake in the analysis, or that there is a sizable systematic error not yet corrected. Perhaps the measurement was done at a much higher temperature than the standard temperature. So this means that you have to go back and discover what could have gone wrong.

You may be tempted to simply increase the size of your quoted error, say to 10 m/s in this example. In that case a discrepancy of 14 m/s would not be terribly significant. However, you have to be careful about doing this. A correct error analysis should give you a correct error size, and the lab instructor will have a pretty good idea of what those error sizes are. More than likely, a discrepancy significantly outside of your error bounds is probably an indication of bad data or faulty analysis.

Comparison of Two Measured Numbers

Proving a Law in Physics

Often times the lab measurement will not be to determine a particular value, such as g, but rather to prove a law in physics such as conservation of momentum. For example, you might be measuring the initial momentum and the final momentum separately in a collision experiment. These are shown in the following table where the units are taken to be kg-m/s:

Experiment on Momentum Conservation I

Initial momentum p	Final momentum p'
1.49 $\pm$0.04	1.56 $\pm$0.06
2.10 $\pm$0.04	2.12 $\pm$0.06
1.16 $\pm$0.04	1.05 $\pm$0.06

How do well do these data confirm the conservation of momentum law? The best way to answer that question is to calculate the *difference* in the final and initial momentum $\Delta p = p - p'$, and see how close that difference is equal to 0. This is achieved by adding a third column to the table:

Experiment on Momentum Conservation II

Initial momentum p	Final momentum p'	$\Delta p = p - p'$
1.49 $\pm$0.04	1.56 $\pm$0.06	-0.07 ± 0.07
2.10 $\pm$0.04	2.12 $\pm$0.06	-0.02 ± 0.07
1.16 $\pm$0.04	1.05 $\pm$0.06	$+0.11 \pm 0.07$

How did we get the errors assigned to the differences in each row? Your first thought (which this Chapter 2 agrees with) is to simply add the errors in each term which would give an error of $\pm$0.10. However, the more mathematically exact method is to *add the errors in quadrature*. In other words, if there is a quantity q which is the difference of two independent quantities x and y each with their measurement error δx and δy, the the error δq is given by

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$$

So in the above table, the momentum differences are consistent with 0 according to the errors quoted.

CHAPTER 2: Fractional Uncertainties

Absolute vs Fractional Uncertainties

We have seen that the correct reporting of a physical measurement requires that one write the "best" value with a quoted error uncertainty. In general then, we write

$$\text{Measured } x = x_{\text{best}} \pm \delta x$$

In this case δx is the absolute uncertainty of the measurement. However, it is often more clear to write the *fractional uncertainty* of the measurement instead of the absolute uncertainty. The idea is that a measurement with a relatively large fractional uncertainty is not as meaningful as a measurement with a relatively small fractional uncertainty.

Definition of Fractional Uncertainty

The fractional uncertainty is just the ratio of the absolute uncertainty, δx to the best value x_{best}:

$$\text{Fractional Uncertainty } \equiv \frac{\delta x}{x_{\text{best}}}$$

In general, the absolute uncertainty δx will be numerically less than the measured best value x_{best}. Otherwise the measurement is generally not worth reporting. The only exception to this rule is in the case where one is trying to make a so-called "null measurement". In that case only the absolute uncertainty has meaning.

For all non-null measurements which have their absolute uncertainties less than the measured quantity itself, then it is standard practice to quote the fractional uncertainty as a percentage. For example, suppose one measures a length l as 50 cm with an uncertainty of 1 cm. Then the absolute quote is

$$l = 50 \pm 1 \text{ cm}$$

while the fractional uncertainty is

$$\text{Fractional Uncertainty } = \frac{\delta l}{l} = \frac{1}{50} = 0.02$$

So the result can also be given as

$$l = 50 \text{ cm} \pm 2\%$$

Multiplying Two Measured Numbers

Another Way of Using the Fractional Uncertainty

The most important application of fractional uncertainties involves their use in deriving the uncertainty of a measurement involving the product of two other measurements. For example, linear momentum p is the product of mass m and velocity v. If we measure the mass to some accuracy and then measure the velocity to some accuracy, what is the accuracy of our momentum measurement? Before answering that question, we return to our general result of a measurement of a quantity x with an uncertainty δx

$$\text{Measured } x = x_{best} \pm \delta x$$

This result can also be written as

$$x = x_{best}(1 \pm \frac{\delta x}{|x_{best}|})$$

So if the fractional uncertainty is 3% then we can write

$$x = x_{best}(1 \pm 0.03)$$

or we can write

$$0.97x_{best} \leq x \leq 1.03x_{best}$$

The best value of the momentum product mv

We now return to the uncertainty of the momentum product mv. We have a best value for the mass m_{best} and a best value for the velocity v_{best}. So the best value for the momentum is just the product of the best values for m and v

$$p_{best} = m_{best}v_{best}$$

The question in now what do we quote for δp

The Uncertainty of a Product

Component uncertainties

The product $p = mv$ has two measured components m and v each with their own uncertainties

$$m = m_{best} \pm \delta m \Longrightarrow m = m_{best}(1 \pm \frac{\delta m}{|m_{best}|})$$

$$v = v_{best} \pm \delta v \Longrightarrow v = v_{best}(1 \pm \frac{\delta v}{|v_{best}|})$$

Now the largest value of p will occur when we take the positive sign of the error in each of the two components m and v.

$$\text{Largest } p = m_{best}(1 + \frac{\delta m}{|m_{best}|})v_{best}(1 + \frac{\delta v}{|v_{best}|})$$

Multiplying out all the terms in parenthesis we get

$$(1 + \frac{\delta m}{|m_{best}|})v_{best}(1 + \frac{\delta v}{|v_{best}|} = 1 + \frac{\delta m}{|m_{best}|} + \frac{\delta v}{|v_{best}|} + \frac{\delta m}{|m_{best}|}\frac{\delta v}{|v_{best}|}$$

Now we assume that the fractional uncertainties are relatively small such that the last term above can be ignored. This then gives us the expected largest value of p

$$\text{Largest } p = m_{best}v_{best}(1 + \frac{\delta m}{|m_{best}|} + \frac{\delta v}{|v_{best}|})$$

Clearly, there will be a similar expression with negative signs for the smalles value of p. We can now get the value of δp in terms of δm and δv. We first write the general expression

$$p = p_{best}(1 \pm \frac{\delta p}{|p_{best}|})$$

We then equate terms in this equation to the previous equation. This leads us to the specific result here

$$\frac{\delta p}{|p_{best}|} = \frac{\delta m}{|m_{best}|} + \frac{\delta v}{|v_{best}|}$$

Here we have a very important, *but provisional*, general rule
The fractional uncertainty of a product is the sum of the fractional uncertainties of the component terms

Please note this is a *provisional* rule. The more exact rule is that the fractional uncertainty of a product is the square root of the sum of the squares of the component fractional uncertainties. The exact rules is compatible with the 2/3 estimate of how many independent measurements will find a value consistent with the value being reported.

The Uncertainty of a Sum

Provisional Sum Rule

We can go through the same exercise to figure out the uncertainty in a sum. Suppose we have a quantity q which is the sum of two other quantities x and y

$$q = x + y$$

Now x and y have uncertainties δx and δy, respectively.
The *provisional* rule is that the uncertainty δq of the sum is the sum of the uncertainties of the component terms:

$$\delta q = \delta x + \delta y$$

This shows up on page 23 as *Provisional Rule 2.18*
The **exact** rule is that the uncertainty in a sum is the square root of the sum of the squares of the uncertainties of the component terms.

Provisional Difference Rule

The uncertainty of a difference is computed the same as the uncertainty of a sum:

$$q = x - y$$

Provisional rule for a difference

$$\delta q = \delta x + \delta y$$

Again the **exact** rule is that the uncertainty in a difference is the square root of the sum of the squares of the uncertainties of the component terms.