Fourier Analysis

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1 Superimposed Waves

Many natural phenomena occur in the form of sinusoidal waves. For example, sound is the oscillation of air density, and light is the oscillation of electric and magnetic fields. When two or more waves of the same type meet at the same point in space, the waves overlap and interfere with one another constructively or destructively. The resultant wave is the sum of the component waves, which is, of course, time dependent. By using an infinite number of waves which relate to each other in specific ways, any shape of periodic waveform can be obtained. Such was the discovery of the famous mathematician, Fourier[1], who states that a function "having a spatial period λ, can be synthesized by a sum of harmonic functions whose wavelengths are integral sub-multiples of λ". Fourier analysis is the process of determining the frequency and amplitude of the harmonic components of a complex wave. Quoting, "Trigonometric series of the form

$$a_0/2 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots \tag{1}$$

are required in the treatment of many physical problems, for example, in the theory of sound, heat conduction, electromagnetic waves, electric circuits, and mechanical vibrations. An important advantage of the series Eq. 1 is that it can represent discontinuous functions, .." [2] The coefficients for $n \geq 0$ are obtained from

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \tag{3}$$

The formulas 2 and 3 are called the *Euler-Fourier* formulas, and the series which results when a_n and b_n are determined by the Euler-Fourier formulas is called the *Fourier Series* of *f(x)* . [3]

Arguments about the properties of $f(x)$ and who first discovered the coefficients have gone on for two centuries. The history involves the names of many famous mathematicians : Euler, d'Alembert, D. Bernoulli, Lagrange, Poisson, Dirichlet, Cauchy, and Riemann. In the twentieth century, work was still being done by Lebesgue. Fourier was working on problems involving heat flow, and the functions f he worked with were solutions to the problems he was interested in. But his results were general enough that he gets his name on the Series. Ignoring the wrangling of the mathematicians, physicists find Fourier Series (and by extension, the integral transforms that also bear his name) useful.

Harmonic waves with appropriate amplitudes can be added to produce complex wave forms. Conversely, frequency dependent elements can be used to determine the harmonic content of complex waves. In this experiment we will first synthesize square and triangular waves and then decompose or analyze them. We will use electric waves in a circuit and analyze them with RC circuits.

[1]Fourier, Jean Baptiste Joseph (1758-1830), French mathematician and politician, son of a tailor and, at age eight, an orphan. One of the savants who accompanied Bonaparte to Egypt in 1798. He was for a time governor of virtually half of Egypt. Published in 1822 *La Theorie analytique de la chaleur*, which included his work for the prize of the Academie des Sciences, which the Academy had refused to publish.

[2]Ref. 1, p.1

[3]*ibid*, p. 176

2 Apparatus

1 Oscilloscope
1 Fourier synthesized, model 9307 Pasco Scientific
1 waveform analyzer model 9302 Pasco Scientific
1 waveform generator
2 banana plug connector wires
2 banana plug to coax cable
1 digital multimeter
capacitor and resistors

3 Procedure

1. Build a square wave by adjusting harmonics.
   Turn all amplification and phase control knobs to the "off" or "zero" positions. Connect the output of the Fourier synthesizer to the oscilloscope using coaxial cable, and turn on both instruments. Set the controls on the synthesizer to produce a sine wave from the first harmonic, and turn the switch of the summing amplifier to the "in" position, so that the signal will reach the output. Connect the trigger output on the synthesizer to the trigger input on the oscilloscope, and set its triggering source to "external". Adjust the oscilloscope until a stable sinusoidal waveform is clearly displayed on the CRT screen. Add in the third harmonic by turning the switch below the control knobs to "in", and adjust the phase and amplitude of the harmonic until the maximum of the first harmonic is overlapped by the minimum of the third harmonic. Add and adjust the fifth harmonic until the peaks of the oscilloscope trace are more flattened. Do likewise with the seventh harmonic. If necessary, readjust the amplitudes of all the harmonics until the oscilloscope trace closely resembles a square wave. Record the amplitude of each harmonic.

2. Build a square wave with computed harmonics.
   Repeat the process of step 1, this time adjusting the amplitude and phase of each harmonic by connecting the oscilloscope and AC voltmeter to the connection of each harmonic. With the oscilloscope at high sensitivity, adjust the phase for each harmonic so that the zero-crossing of each harmonic is time aligned, typically using the vertical line on the oscilloscope as a reference. This will ensure that each harmonic is a sine wave with the same phase as the other harmonics. Adjust the amplitude of the nth harmonic, V_n, to be proportional to $1/n$. Once the harmonics are individually adjusted, look at their sum.

3. Analyze components in a square wave.
   Connect the output of the Fourier synthesizer to the input of the waveform analyzer, using the banana plug connector wires, and connect the oscilloscope to the output of the waveform analyzer. Set the analyzer so that it allows a certain band of frequencies to pass through and be amplified, Increase the gain on the analyzer until the signal is clearly locked in by the oscilloscope, but not so much that the peaks of the signal become flattened, indicating saturation of the amplifier. Set the multiplier knob at $\times 10$, while watching the oscilloscope screen. When a sinusoidal waveform is shown on the screen at its maximum amplitude, record its frequency. Repeat the process using the frequency range $\times 100$ on the multiplier, and again with the range of $\times 1000$.

   Note: The signals found at the upper frequency levels in the $\times 1000$ mode should not be included in the Fourier analysis, since they are a result of the internal circuitry and are not produced by the synthesizer itself.

4. Next create a triangular wave using the third, fifth and seventh harmonics in conjunction with the primary sine wave. Adjust the phase of the third harmonic so that every other maximum overlaps the maximum to the first. Add in the fifth harmonic and adjust the phase so that the trace on the oscilloscope more closely resembles that of a triangle wave. Finally add in the seventh harmonic to finish the construction. If necessary, readjust the amplitude and phase of each harmonic to obtain good results. Record the amplitude of each harmonic.

5. Repeat step 4, but this time after adjusting the phase of each harmonic to that the first *peak* is aligned, i.e., the waves are all cosines with the same phase, and so that $V_n \propto 1/n^2$.

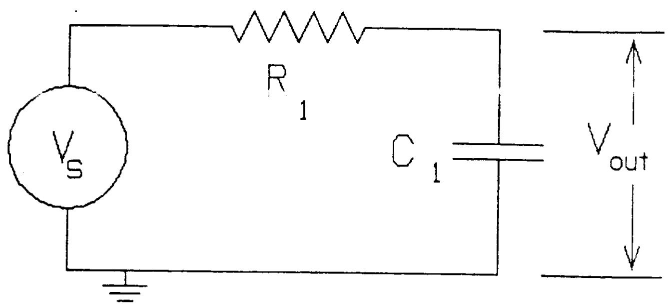

Figure 1: Circuit diagram for a low-pass filter. V_s indicates the input signal

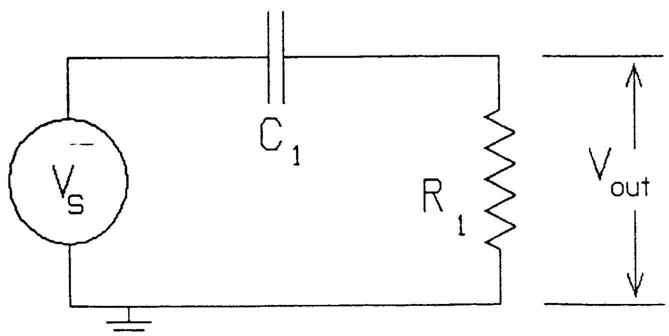

Figure 2: Circuit diagram for a high-pass filter. V_s indicates the input signal

6. We will use a simple RC circuit and will examine how this circuit can be used to modify (or filter) the frequency characteristics of a signal passed through it. Construct the low-pass filter circuit as shown in Fig. 1 using 225 Ω resistor for R and a 0.47 μF for C. Use a multimeter to measure the value of the resistor, as it may happen that you have a different one in your lab set-up. Our goal will be to measure the gain curve of the filter: i.e. we would like to construct a graph of the ratio of the output-to-input voltage on the filter as a function of the frequency of the input signal. Since this is a low-pass filter, we expect that the gain will be close to unity for low frequencies and it will gradually reduce for high frequencies. The slope of the gain curve will depend on the value of the RC components that we use.We need to measure simultaneously the amplitude of the input and the output voltage. Put a "T" split at the output of the wave generator and send one copy of the signal to channel 1 of the scope and another one, to the input of the filter. Use the scope probe to connect the output from the filter to channel 2 of the scope. Use the scope measurement functions to measure the amplitude and frequency in both channels. Vary the frequency of the input signal and observe how the amplitude of the filter output varies with frequency. When you get a feeling of how fast the output amplitude changes with frequency, select a good set of 10 points for which you will record V_{in}, V_{out}, f. Next construct the gain curve: $g(f) = V_{out}/V_{in}$.

7. Repeat step 6 for the high-pass filter.

4 Results

1. Can a square wave be created using different harmonics other than the first, third, fifth and seventh? What effect does adjusting the phase of each harmonic have upon the composite waveform?

2. After analyzing the composite wave, what is the frequency of the fundamental wave? The third harmonic? The fifth and seventh?

3. Make a bar graph of amplitude vs. harmonic frequency for both the square and the triangle wave. Qualitatively, what is the difference in phase, amplitude, and harmonics of a triangle wave as opposed to a square wave?

5 References

1. I. S. Sokolnikoff and R. M. Redheffer, *Mathematics of Physics and Modern Engineering*, (New York, McGraw-Hill, 1958)

2. A. P French, *Vibrations and Waves*, (New York: W. W. Norton Company, Inc., 1971)

3. Douglas C. Giancoli, *General Physics*, (Englewood Cliffs: Prentice-Hall, Inc. 1984)

4. Hecht, *Modern Optics*